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RADIATION FROM SOURCES IN MAGNETOPLASMA WITH A SEPARATION BOUNDARY

By

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RADIATION FROM SOURCES IN MAGNETOPLASMA WITH A SEPARATION BOUNDARY

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Abstract This dissertation deals with a class of problems of electromagnetic radiation from elementary sources in the presence of an anisotropic plasma half-space. All of the problems considered have the following common features: (a) the principal axes of the sources are always parallel to the interface, and (b) the steady magnetic field is also always parallel to the interface and either normal to or in the direction of the principal axis of the source.

The work is divided in three parts. In Part I we are concerned with the fields of a horizontal magnetic dipole in and out of the magnetoplasma when the steady magnetic field is along the axis of the dipole. The rigorous formulation of the problem is carried out to the point where the determination of the pertinent boundary coefficients remains to be a straight-forward but not simple algebraic process. Due to the prohibitive algebraic complexity involved in the explicit finding of the necessary boundary coefficients, the high frequency approximation is introduced at that point and the approximate form of the boundary coefficients are subsequently found. The field integrals are evaluated in the air-region for the condition when the observation point is at a large distance from the source. Several numerical examples pertaining to the magnetic dipole in the ionosphere are presented depicting in graph form the effects of the steady magnetic field on the radiation

pattern under several conditions of wave frequency and the depth of the source burial.

Part II is devoted to the problems of the electric current line source. The case of the steady magnetic field normal to the line source is rigorously formulated and the field integrals are evaluated with no restriction on the wave frequency or the magnitude of the steady magnetic field. The subject of the laterial waves is elaborated extensively and the graphs of the radiation field patterns are presented for the case when the source is immersed in the lower edges of the ionosphere for frequencies ranging from VLF to HF. The latter portion of Part II is devoted to the formulation and solution of the problem when the steady magnetic field is along the line source. In this case it is found that the steady magnetic field has no effect on the radiation field which could also be predicted by physical reasoning.

Finally, Part III is devoted to the problems of the magnetic current line sources. The case of the steady magnetic field normal to the line source is rigorously formulated and the field integral evaluated with no restriction on the wave frequency or the magnitude of the steady magnetic field. This case is analogous to the corresponding case of Part II and many conclusions drawn in that part also apply here. The latter portion of Part III is devoted to the formulation and solution of the case when the steady magnetic field is along the line source. Unlike the corresponding case of the electric line source in Part II, it is found in this case that the radiation field is affected by the action of the steady magnetic field which manifests itself in the lack of symmetry of the radiation pattern.

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CHAPTER 1

INTRODUCTION

This dissertation deals with a class of problems of electromagnetic radiation from elementary sources in the presence of an anisotropic plasma half-space. All of the problems considered have the following common features: (a) the principal axes of the sources are always parallel to the interface, and (b) the steady magnetic field is also always parallel to the interface and either normal to or in the direction of the principal axis of the source.

1.1 OBJECTIVES AND THE STATEMENT OF THE PROBLEM

We shall set forth the following as objectives of this dissertation. We shall attempt to understand and explain analytically the action and the effects produced by a steady magnetic field on the radiation of electromagnetic waves from sources in the presence of a magnetoplasma half-space. Consistent with these objectives, we shall formulate and solve certain basic boundary value problems of electromagnetic wave propagation whose isotropic equivalents are well-known. By comparing our solutions with the corresponding solutions dealing with isotropic half-spaces, we shall be able to draw conclusions about effects produced by the steady magnetic field.

Having thus decided on the basic objectives of this thesis, we are now faced with the definition of the problem. In order to have as complete a picture as possible about the effects of the steady magnetic field on the

various wave polarizations and the aspects and directions of the steady magnetic field, it was decided in the early stages of this research to investigate the fields produced by a horizontal magnetic dipole antenna with the steady magnetic field in the direction of the dipole. The choice of a horizontal magnetic dipole as the source of the electromagnetic waves was motivated by the fact that such an antenna in the presence of an interface effectively produces both vertically and horizontally polarized waves (20), thus, both wave polarizations could be studied in a single problem. Moreover, since the electric field lines form circular loops about the axis of a magnetic dipole and the steady magnetic field is along the axis of the same, then the electric field is normal to the steady magnetic field and, thus, maximum interaction of the electromagnetic wave with the steady magnetic field could be expected.

In the process of the investigation into the solution of a magnetic dipole problem, it will soon become evident that a rigorous solution, if at all possible, would be extremely complex mathematically and the physics of the problem necessarily obscure and difficult to interpret. It will become apparent, however, that certain useful information could still be obtained from the formulation of this problem at the high frequency limit where the ratio of the cyclotron frequency to wave frequency is small. We shall exploit this possibility a great deal and indeed we shall obtain an approximate solution to the horizontal magnetic dipole in or out of the magnetoplasma valid over the entire hemisphere excepting a small region about a critical angle corresponding to the angle of total internal reflection in the plasma in the absence of the steady magnetic field.

Since the investigation of the vertically and horizontally polarized waves as produced by a horizontal dipole is not expedient, we shall look into the possibility of studying these two polarizations separately. To this end

we propose to investigate the solutions to the following two-dimensional problems:

(a) the electric current line source with the steady magnetic field normal,

(b) the electric current line source with the steady magnetic field parallel,

(c) the magnetic current line source with the steady magnetic field normal,

(d) the magnetic current line source with the steady magnetic field parallel.

In the plane normal to the axis of an electric or a magnetic dipole, the resulting fields are identical in form, except for the space dependence, to the corresponding cases of the line sources. Thus, by solving the above two-dimensional problems, we obtain correct expressions for the dipole fields in a plane normal to the dipole axis for the following cases of the dipole-steady magnetic field configurations:

(a) steady magnetic field normal to an electric dipole,

(b) steady magnetic field parallel to an electric dipole,

(c) steady magnetic field normal to a magnetic dipole,

(d) steady magnetic field parallel to a magnetic dipole.

In what follows, we shall have the opportunity to study the electromagnetic field structure due to each one of the excitations proposed. In particular, we will be able to elaborate extensively on the salient features produced by the steady magnetic field in both the radiation and lateral fields.

1.2 HISTORICAL BACKGROUND AND RELATED PROBLEMS

The problem of radiation from sources in the presence of isotropic interfaces is one of the oldest and most honored boundary value problems in the history of electromagnetic theory. It originated with Sommerfeld (20), who in 1909 published the formulation and the solution to the problem of an

oscillating vertical dipole in the presence of a conducting, homogeneous and isotropic half-space. Later, in 1926, Sommerfeld formulated and solved the problems of horizontal electric and magnetic dipole in the presence of a conducting, homogeneous and isotropic half-space. Since that time a great deal of effort has been expanded by various workers in the field, mainly on the subject of approximate evaluation of the definite integrals that appeared in Sommerfeld's solution. Of all these works, the writer found the work of Banos and Wesley (2,3) to be the most comprehensive one.

The phenomenon of lateral waves, to which we shall have the occasion to refer a great deal in this thesis, was first reported by Schmidt (19) in 1938. He performed a series of experiments with pressure waves produced by a spark discharge in the presence of an interface between two liquid half-spaces of sodium chloride (sound velocity about 1600 meters per second) and xylol (sound velocity 1175 meters per second.) The series of schlieren-photographs obtained by him clearly exhibit the mechanism by which these lateral waves are produced. Subsequently, in 1939, Joos and Teltow (12) pointed out that the transient phenomena observed by Schmidt can be readily carried over to the stationary fields in electrodynamics and as such, these phenomena are indeed already included in the original Sommerfeld's solution (20). Finally, in 1942, Ott (18) reviewed Sommerfeld's work and rigorously showed how the lateral field follows from that solution. Later, in 1953, Gerjuoy (8) also elaborated on the same subject. A very comprehensive treatment of the subject of the lateral waves may be found in Brekhovskikh book (5).

The subject of electromagnetic radiation from sources located in magnetoplasma did not, surprisingly enough, attract any attention for a long time. It was not until 1957 that the first paper on this subject was published by Bunkin (6). He found a general solution to the problem of the field produced by a given distribution of external currents in an infinite

homogeneous medium having arbitrary anisotropy. For the case of a magneto-active medium, he found the multipole expansion of the radiation field and discussed some aspects of the solution to the problem of an oscillating electric dipole. He found explicit expressions for the dipole radiation field only when the observation point was either on the dipole axis or on a line normal to it, while the dipole itself was either along the direction of the steady magnetic field or perpendicular to it. The nature of the difficulties that prevented Bunkin from obtaining complete expressions for the radiation fields was the fact that the saddle point which is given by a solution to a transcendental equation could not, in general, be evaluated explicitly.

The same problem was then considered by other workers. Kogelnik (13) formulated the electric dipole problem by a somewhat different method and found the expression for the total power radiated in a lossless plasma for the cases when a single dipole is oriented along or normal to the steady magnetic field and two crossed dipoles in a plane normal to the same. Arbel (1), Kuehl (14), and Meecham (17) also considered the same problem.

The problem of radiation in free space from sources immersed in a magnetoplasma half-space was first considered by Barsukov (4) in 1959. He obtained expressions for the Poynting vector in the air in the case when the steady magnetic field was normal to the boundary and an electric dipole was either normal or parallel to the boundary. He also presented a numerical example for a vertical dipole at the boundary and for the particular set of conditions considered, he found that the plasma anisotropy manifested itself strongly in the directivity of the radiation pattern.

Finally, we note that the problem of a vertical dipole immersed in a magnetoplasma half-space with the steady magnetic field normal to the boundary was also considered by Arbel (1) in 1960.

1.3 CONTRIBUTIONS OF THIS THESIS

In this work we treat for the first time the problems of electromagnetic radiation from sources in the presence of a magnetoplasma half-space when the steady magnetic field is parallel to the separation boundary. We investigate in detail the effects of the plasma anisotropy on both vertically and horizontally polarized electromagnetic waves.

We also, for the first time, elaborate on the subject of the lateral waves in connection with the magnetoplasma interfaces. In particular, we find that such waves are induced under most of the conditions and they form an important contribution to the total field especially in the proximity of the interface.

1.4 THE METHOD OF ATTACK

Unlike in the previous work on a related subject (4), we shall use in this thesis the powerful and rigorous method of Fourier transforms in Cartesian coordinates to formulate the problems. Such representation necessarily yields a complete and unique solution and furthermore after two integrations, allows unequivocal determination of the path of integration in the complex plane of the third transform variable which must lie within a certain strip of analyticity (23, p. 44.)

We divide this thesis in three parts. In Part I, which includes Chapters 2, 3, 4, 5, 6 and 7, we are concerned with the problem of a horizontal magnetic dipole in and out of the magnetoplasma when the steady magnetic field is along the axis of the dipole. The rigorous formulation of the problem is carried out to the point where the determination of the pertinent boundary coefficients remains to be a straight-forward, but not simple, algebraic process. Due to the prohibitive algebraic complexity involved in the explicit

finding of the necessary boundary coefficients, the high frequency approximation is introduced at that point and the boundary coefficients are subsequently found. The field integrals are evaluated for the air-region for the condition when the observation point is at a large distance from the source. Several numerical examples pertaining to the magnetic dipole in the ionosphere are presented depicting in graph form the effects of the steady magnetic field on the radiation pattern under several conditions of the wave frequency and the depth of the source burial.

Part II which consists of Chapters 8, 9, and 10, is devoted to the problems of the electric current line sources. The case of the steady magnetic field normal to the line source is rigorously formulated in Chapter 8 and the integrals are evaluated in Chapter 9 with no restrictions on frequency. The subject of lateral waves is elaborated on extensively and the graphs of the radiation field pattern are presented for the case when the source is immersed in the lower edges of the ionosphere for frequencies ranging from VLF to HF. Chapter 10 is devoted to the formulation and solution of the problem when the steady magnetic field is along the line source. In this case it is found that the steady magnetic field has no effect on the radiation field which could also be predicted by physical reasoning.

Finally, Part III consisting of Chapters 11, 12, and 13, is devoted to the problems of the magnetic current line sources. The case of the steady magnetic field normal to the line source is rigorously formulated in Chapter 11 and the integrals are evaluated in Chapter 12 with no restriction on frequency. This case is analogous to the corresponding case of Part II and many conclusions drawn in that part apply here also. Chapter 13 is devoted to the formulation and solution of the case when the steady magnetic field is along the line source. Unlike the corresponding case of the electric line source in Part II, it is found that here the radiation field is affected by the

action of the steady magnetic field which manifests itself in the lack of symmetry of the radiation pattern.

PART I

RADIATION FROM A HORIZONTAL MAGNETIC DIPOLE IN THE PRESENCE OF A MAGNETOPLASMA HALF-SPACE

CHAPTER 2

RIGOROUS FORMULATION OF THE PROBLEM FOR A MAGNETIC DIPOLE SOURCE

In this chapter we shall be concerned with the finding of appropriate integral representations for the Cartesian components of the field vectors for the magnetoplasma- and the air-half- spaces. This will entail the description of the magnetoplasma's properties on the macroscopic scale, review of the fundamental field equations and the definition of the source of electromagnetic waves.

2.1 STATEMENT OF THE PROBLEM

The geometry of the problem is shown in Figure 2.1. The horizontal plane $z = 0$ coincides with the interface between the anisotropic homogeneous plasma and air. For convenience we shall call the plasma medium (1) and the air medium (0). The plasma, in addition to its anisotropy, may be lossy while the conductivity of the air is zero. Moreover, we assume that both media have the same magnetic inductive capacity of free space, μ_0. The factor causing the anisotropy of the plasma, the steady magnetic field H_{DC}, is oriented in the positive x-direction as well as the source of the electromagnetic waves, the magnetic dipole.

Since we shall be mainly interested in the fields everywhere in the air (the upper hemisphere), the final results will be expressed in spherical coordinates. To this end we define the spherical angles φ and Θ as shown in

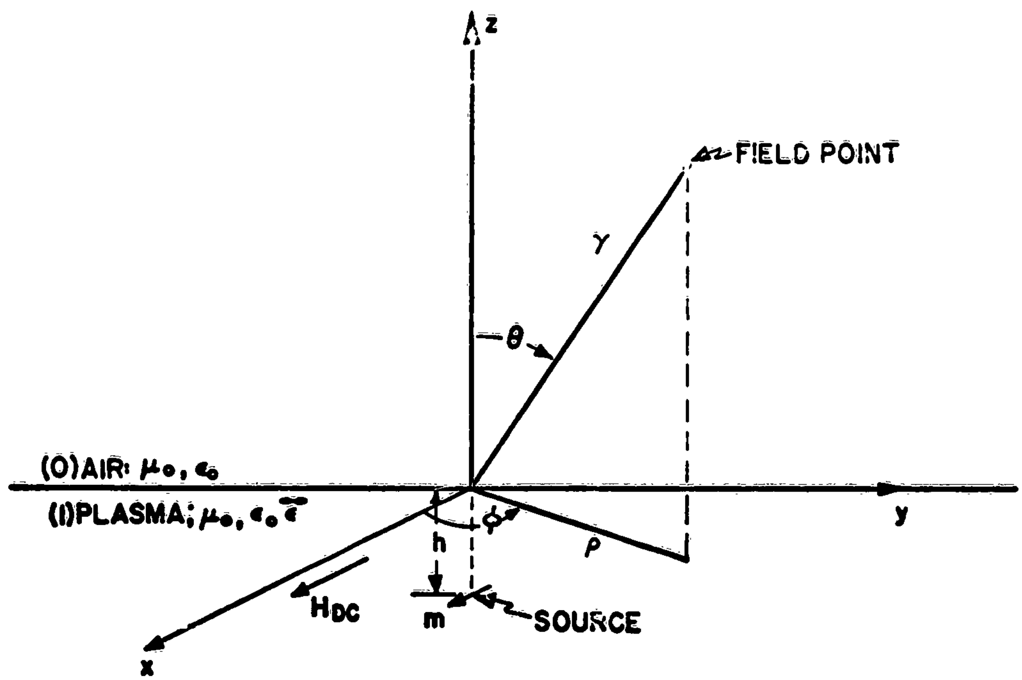

Figure 2.1 - Geometry of the Problem of a Magnetic Dipole in Magnetoplasma

tric field, the electrons follow the direction of the electric field between the collisions with the other constituents of the gas. A single collision can be pictured as a sudden reduction of the momentum $m\vec{u}$ of a free electron to zero. This change in the momentum of the electron takes place within an average time interval ν^{-1}, ν being the collision frequency. Thus, the collision effects are equivalent to an additional force $-m\nu\vec{u}$ and the total force exerted on the electron becomes

$$\dot{\vec{u}} + \nu\vec{u} = -\frac{|e|}{m}\left(\vec{E}e^{-i\omega t} + \vec{u} \times \vec{B}_{DC}\right) \tag{2.1}$$

where the field of the wave is assumed to vary as $e^{-i\omega t}$. In writing (2.1) we assumed that the magnitude of the magnetostatic field will always be much greater than the magnitude of the alternating magnetic field so that the latter one can be neglected.

If there is N electrons per unit volume, then the associated conduction current density $\vec{J}_c$ is given by

$$\vec{J}_c\, e^{-i\omega t} = -\,|e|\, N\vec{u} \tag{2.2}$$

assuming that all electrons have the same velocity. (Otherwise, we would have to integrate over the velocity distribution.) Combining (2.1) and (2.2) one obtains

$$\vec{J}_c = i\omega\epsilon_o\left\{ p^2 \left[-i\sigma\vec{1}_o \times +(1+iq)\,(\delta_{ij})\right]^{-1} \right\} \vec{E} \tag{2.3}$$

where $\vec{1}_o$ is a unit vector in the direction of the steady magnetic field, δ_{ij} is the Kronecker delta and

$$p = \frac{\omega_p}{\omega} = \frac{|e|}{\omega}\sqrt{\frac{N}{m\epsilon_o}}$$

$$q = \frac{\nu}{\omega} \tag{2.4}$$

$$\sigma = \frac{\omega_H}{\omega} = -\frac{|e|\,B_{DC}}{\omega\, m}\ .$$

The total current in the plasma $\vec{J}_T$ is the sum of the conduction current $\vec{J}_c$ and the displacement current $\vec{J}_D = -\,i\,\omega\,\epsilon_o\vec{E}$. Thus we write

Figure 2.1.

2.2 FUNDAMENTAL EQUATIONS

The definition of the present two-media problem implies the solution to Maxwell's equation subject to the usual boundary conditions at the interface and proper behavior at infinity. In the air-region it is convenient to introduce the Hertzian vector potential of the magnetic type whereas in the plasma-region it is found to be more convenient to work with actual field components. The imposition of the requirement of the radiation condition at infinity in addition to the boundary condition at the interface furnishes the necessary relationships which render the solution determinate and unique.

2.2a Magnetoplasma description--The term "plasma" describes an electrically neutral gaseous medium consisting of an equal number of free electrons and positive ions. We shall use the term "magnetoplasma" to describe plasma with an impressed magnetostatic field.

In the absence of any external electric or magnetic fields, the motion of the free electrons and ions is random because it is caused by the thermal agitation alone. Since the mass of an ion is much greater than that of the electron, the velocity of the former is much lower than that of the latter. Thus when the wave frequency is high enough, the current contribution due to the motion of the positive ions becomes negligible. In the presence of a steady magnetic field H_{DC}, the electrons describe spiral paths between collisions about the lines of force. This motion is uniform and its projection on a plane normal to the line of force is a uniform circular motion with an angular velocity corresponding to the cyclotron frequency $\omega_H = e B_{DC}/m$, where $e = -1.6 \times 10^{-19}$ Coulomb is the electronic charge and $m = 9.1 \times 10^{-31}$ kilogram is the electronic mass. If now, in addition, there is a time varying elec-

$$\vec{J}_T = -i\omega\epsilon_0\left\{ (\delta_{ij}) + p^2\left[i\sigma\vec{T}_0 \times -(1+iq)(\delta_{ij})\right]^{-1} \right\} \vec{E} = -i\omega\epsilon_0\vec{\epsilon}\vec{E} \,. \tag{2.5}$$

We shall henceforth call $(\vec{\epsilon})$ the permittivity tensor of the magnetoplasma (24, p. 297.) One can, for convenience, also introduce the electric displacement vector defined by

$$\vec{D} = \epsilon_0 \vec{\epsilon}\ \vec{E} \,. \tag{2.6}$$

Since the medium is assumed to be electrically neutral (there are as many negative charges as there are positive charges), we note that

$$\vec{\nabla} \cdot \vec{D} = 0. \tag{2.7}$$

The components of the permittivity tensor which we shall need later have been found elsewhere (24, p. 297) for any orientation of the steady magnetic field H_{DC} . When the steady magnetic field is oriented in the positive x-direction, which is the case under consideration, the tensor $(\vec{\epsilon})$ is

$$\vec{\epsilon} = \epsilon \begin{bmatrix} \frac{\zeta}{\epsilon} & 0 & 0 \\ 0 & 1 & i\kappa \\ 0 & -i\kappa & 1 \end{bmatrix} \,. \tag{2.8}$$

Since $\vec{\epsilon}$ is non-singular, its inverse can be found to have the form

$$\vec{\epsilon}^{-1} = \frac{1}{\chi} \begin{bmatrix} \chi & 0 & 0 \\ 0 & 1 & -i\kappa \\ 0 & i\kappa & 1 \end{bmatrix} \tag{2.9}$$

where

$$\chi = \frac{\epsilon^2 - \eta^2}{\epsilon} \qquad \kappa = \frac{\eta}{\epsilon} \tag{2.10}$$

and

$$\zeta = \zeta' + i\zeta'' \qquad \epsilon = \epsilon' + i\epsilon'' \qquad \eta = \eta' + i\eta'' \tag{2.11}$$

Moreover

$$\left.\begin{aligned} \zeta' &= 1 - \frac{p^2}{1+q^2} \\ \zeta'' &= \frac{q\,p^2}{1+q^2} \end{aligned}\right\} \quad (2.12a)$$

$$\left.\begin{aligned} \epsilon' &= 1 - \frac{p^2(1-\sigma^2+q^2)}{[(\sigma+1)^2+q^2][(\sigma-1)^2+q^2]} \\ \epsilon'' &= \frac{qp^2(1+\sigma^2+q^2)}{[(\sigma+1)^2+q^2][(\sigma-1)^2+q^2]} \end{aligned}\right\} \quad (2.12b)$$

$$\left.\begin{aligned} \eta' &= \frac{\sigma p^2(\sigma^2-1+q^2)}{[(\sigma+1)^2+q^2][(\sigma-1)^2+q^2]} \\ \eta'' &= \frac{2\sigma q\,p^2}{[(\sigma+1)^2+q^2][(\sigma-1)^2+q^2]} \end{aligned}\right\} \quad (2.12c)$$

When the collisions are small and can be neglected, the above parameters have simpler forms as follows:

$$\zeta = 1 - p^2 \qquad (2.13a)$$

$$\epsilon = 1 - \frac{p^2}{1-\sigma^2} \qquad (2.13b)$$

$$\eta = \frac{-\sigma p^2}{1-\sigma^2} \quad . \qquad (2.13c)$$

Furthermore, when the magnitude of the magnetostatic field is small or the wave frequency is sufficiently high so that

$$|\sigma| = \left|\frac{eB_{oc}}{\omega m}\right| \ll 1 \qquad (2.14)$$

we can write for a lossless magnetoplasma

$$\zeta = 1 - p^2 \qquad (2.15a)$$

$$\epsilon \sim \zeta \qquad (2.15b)$$

$$\eta \sim -\sigma p^2 \quad . \qquad (2.15c)$$

As a consequence, the relations (2.10) can be written

$$\chi \sim \zeta$$

$$\kappa \sim \frac{-\sigma p^2}{1 - p^2} \qquad (2.16)$$

We note that for κ to remain small $p^2 \ll 1$ or, in other words, the wave frequency must be much larger than the plasma frequency.

We shall introduce still a different notation which will prove useful later. Namely, we define an index of refraction of the plasma in absence of the magnetostatic field

$$n^2 = 1 - p^2 \; , \qquad (2.17)$$

in terms of which equations (2.15) and (2.16) become

$$\zeta = n^2 \qquad (2.18a)$$

$$\epsilon \sim \zeta = n^2 \qquad (2.18b)$$

$$\eta \sim -\sigma (1 - n^2) \qquad (2.18c)$$

$$\chi \sim n^2 \qquad (2.18d)$$

$$\kappa \sim -\sigma \left(\frac{1 - n^2}{n^2}\right) \qquad (2.18e)$$

<u>2.2b The nature of the source</u>--For the purpose of the present problem it will be assumed that the source of the electromagnetic waves consists of a small wire loop carrying an alternating current $I e^{-i\omega t}$. Such a small loop can be fed with a standard coaxial cable transmission line with a proper matching section in-between. If the loop is small enough its electromagnetic effects can be adequately represented by its equivalent magnetic dipole moment (7, p. 291)

$$\vec{m} = I \vec{S} e^{-i\omega t} \qquad (2.19)$$

where $\vec{S}$ is the surface vector of area enclosed by the loop. The magnetic dipole moment can, in turn, be thought of as a product of "magnetic current" K and the length of the dipole "l", thus

$$K \, l \cdot \vec{1}_m = \vec{m} \quad . \qquad (2.20)$$

To localize the source properly we write for the "magnetic current density"

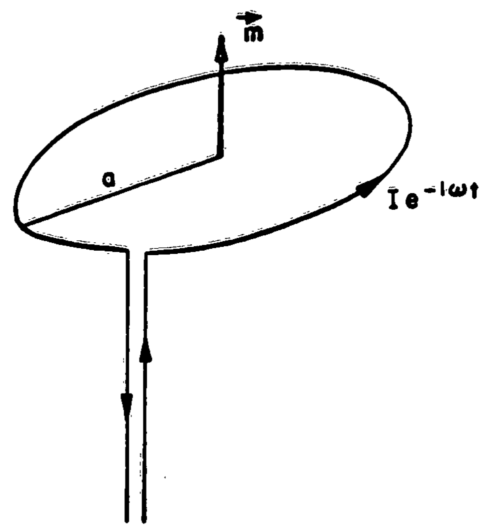

Figure 2.2 - A Small Electric Current Loop and its Equivalent

$$\vec{J}_m = \vec{m}\,\delta(x)\,\delta(y)\,\delta(z+h) \tag{2.21}$$

where δ is the Dirac delta function (7, p. 43).

2.2c The field equations--We have shown that our source of electromagnetic waves may be regarded as a magnetic dipole singularity located at a point (0,0,-h). Then for the magnetoplasma region the appropriate form of Maxwell's equations is

$$\begin{aligned}
\vec{\nabla} \times \vec{E}_1 &= i\omega\mu_0 \vec{H}_1 - m\,\delta(x)\delta(y)\delta(z+h)\,\vec{1}_x \\
\vec{\nabla} \times \vec{H}_1 &= -i\omega\epsilon_0 \overleftrightarrow{\epsilon}\,\vec{E}_1 \\
\vec{\nabla} \cdot \vec{D}_1 &= 0 \\
\vec{\nabla} \cdot \vec{B}_1 &= \rho_m
\end{aligned} \tag{2.22}$$

where ρ_m is the fictitious "magnetic charge density" related to the "magnetic current" by the continuity equation

$$\vec{\nabla} \cdot \vec{J}_m = -\frac{\partial \rho_m}{\partial t} \quad . \tag{2.23}$$

Since $\overleftrightarrow{\epsilon}$ is non-singular the second equation of (2.22) can be written

$$\overleftrightarrow{\epsilon}^{-1}\,\vec{\nabla} \times \vec{H}_1 = -i\omega\epsilon_0 \vec{E}_1 \quad . \tag{2.24}$$

Now we operate with the curl on the last equation and substitute the first equation of (2.22) to obtain

$$-\vec{\nabla} \times \overleftrightarrow{\epsilon}^{-1}\,\vec{\nabla} \times \vec{H}_1 + k_0^2 \vec{H}_1 = -i\omega\epsilon_0 m\,\delta(x)\delta(y)\delta(z+h)\,\vec{1}_x \quad . \tag{2.25}$$

Using the inverse of the permittivity tensor of equation (2.9) and carrying out the necessary algebraic operations, one obtains a set of simultaneous equations as follows:

$$\begin{bmatrix}
\chi k_0^2 + \partial_y^2 + \partial_z^2 & -\partial_x(\partial_y + i\kappa\partial_z) & -\partial_x(-i\kappa\partial_y + \partial_z) \\
-\partial_x(\partial_y - i\kappa\partial_z) & \chi k_0^2 + \partial_x^2 + \frac{\chi}{\xi}\partial_z^2 & -\frac{\chi}{\xi}\partial_y\partial_z - i\kappa\partial_x^2 \\
-\partial_x(i\kappa\partial_y + \partial_z) & -\frac{\chi}{\xi}\partial_y\partial_z + i\kappa\partial_x^2 & \chi k_0^2 + \partial_x^2 + \frac{\chi}{\xi}\partial_y^2
\end{bmatrix} = -i\omega\epsilon_0\chi\,\delta(x)\delta(y)\delta(z+h) \begin{bmatrix} m \\ 0 \\ 0 \end{bmatrix} \tag{2.26}$$

which we shall leave in this form for the time being. From the second equation of (2.22) we note that the electric field can be expressed in terms of the magnetic field as follows:

$$\begin{bmatrix} E_{x_1} \\ E_{y_1} \\ E_{z_1} \end{bmatrix} = \frac{1}{i\omega\epsilon_0\chi} \begin{bmatrix} 0 & \frac{\chi}{\zeta}\partial_z & -\frac{\chi}{\zeta}\partial_y \\ -i\kappa\partial_y - \partial_z & i\kappa\partial_x & \partial_x \\ \partial_y - i\kappa\partial_z & -\partial_x & i\kappa\partial_x \end{bmatrix} \begin{bmatrix} H_{x_1} \\ H_{y_1} \\ H_{z_1} \end{bmatrix} \tag{2.27}$$

In the air-region it will be convenient to use the Hertzian vector potential of the magnetic type defined by[†]

$$\vec{E}_0 = i\omega\mu_0 \vec{\nabla} \times \vec{\Pi}_0 \tag{2.28}$$

and satisfying the homogeneous vector wave equation

$$(\nabla^2 + k_0^2)\vec{\Pi}_0 = 0 \,. \tag{2.29}$$

The magnetic field is expressed in terms of this Hertzian vector as follows:

$$\vec{H}_0 = k_0^2\vec{\Pi}_0 \quad + \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{\Pi}_0) \tag{2.30}$$

2.3 FOURIER INTEGRAL REPRESENTATION IN CARTESIAN COORDINATES

The formulation of the present boundary value problem can be simplified a great deal by expressing the field components in the magnetoplasma and the components of the Hertzian vector in the air in terms of their triple Fourier integral representation in Cartesian coordinates in the transform space as well as in the configuration space. Such representation necessarily yields a complete and unique solution, and furthermore, after performing two integrations, it allows the unequivocal determination of the path of integration in

[†] In the literature on electromagnetic theory the Hertzian vector of the electric type is denoted by $\vec{\Pi}$ and the one of the magnetic type by $\vec{\Pi}^*$ (22,p.29). In this thesis we use only the Hertzian vector of the magnetic type and we shall denote it by $\vec{\Pi}$ reserving (*) to denote the complex conjugate.

the complex plane of the third transform variable which must lie within a certain strip of analyticity (23, p. 44).

Thus, we introduce a triple Fourier transform pair defined by

$$\overset{\approx}{F}(\alpha_1,\alpha_2,\alpha_3)=\frac{1}{(2\pi)^{3/2}}\iiint_{-\infty}^{\infty}F(x,y,z)\,e^{-i(\alpha_1 x+\alpha_2 y+\alpha_3 z)}dx\,dy\,dz \tag{2.31}$$

and

$$F(x,y,z)=\frac{1}{(2\pi)^{3/2}}\iiint_{-\infty}^{\infty}\overset{\approx}{F}(\alpha_1,\alpha_2,\alpha_3)\,e^{i(\alpha_1 x+\alpha_2 y+\alpha_3 z)}d\alpha_1\,d\alpha_2\,d\alpha_3 \tag{2.32}$$

In what follows we shall also need the transforms of the derivatives. These can be obtained by integrating by parts. Thus, for example, consider the integral

$$\begin{aligned}\iiint_{-\infty}^{\infty}\partial_x F\,e^{-i(\alpha_1 x+\alpha_2 y+\alpha_3 z)}dx\,dy\,dz &= \iint_{-\infty}^{\infty}F\,e^{-i\alpha_1 x}\Big|_{x=-\infty}^{x=\infty}e^{-i(\alpha_2 y+\alpha_3 z)}dy\,dz\\ &\quad+i\alpha_1\iiint_{-\infty}^{\infty}F\,e^{-i(\alpha_1 x+\alpha_2 y+\alpha_3 z)}dx\,dy\,dz\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}\iiint_{-\infty}^{\infty}\partial_x^2 F\,e^{-i(\alpha_1 x+\alpha_2 y+\alpha_3 z)}dx\,dy\,dz &= \iint_{-\infty}^{\infty}[\partial_x F+i\alpha_1 F]\,e^{-i\alpha_1 x}\Big|_{x=-\infty}^{x=\infty}e^{-i(\alpha_2 y+\alpha_3 z)}dy\,dz\\ &\quad-\alpha_1^2\iiint_{-\infty}^{\infty}F\,e^{-i(\alpha_1 x+\alpha_2 y+\alpha_3 z)}dx\,dy\,dz\,.\end{aligned} \tag{2.34}$$

The vanishing of the integrated part of equation (2.33) and (2.34) at the upper and lower limits is assured for all real values of the transform variable α_1, providing we obey the radiation condition, i.e., with y and z fixed we should have as a solution

$$F \xrightarrow[|x| \to \infty]{} \frac{e^{ik|x|}}{|x|} \qquad (2.35)$$

for $\mathrm{Im}\{k\} \geqslant 0$.

Thus, assuming that our solution satisfies the radiation condition, we establish the following correspondences:

$$\begin{aligned} \partial_x F &\Longleftrightarrow i\alpha_1 \tilde{\tilde{\tilde{F}}} \\ \partial_y F &\Longleftrightarrow i\alpha_2 \tilde{\tilde{\tilde{F}}} \\ \partial_z F &\Longleftrightarrow i\alpha_3 \tilde{\tilde{\tilde{F}}} \end{aligned} \qquad (2.36)$$

2.3a The particular integral corresponding to the source—To transform the inhomogeneous system of simultaneous equations (2.26), one multiplies both sides by the exp $\{-i(\alpha_1 x+\alpha_2 y+\alpha_3 z)\}$ and integrates with respect to the real variables x, y, and z between $-\infty$ and $+\infty$. The right-hand side of (2.26) yields at once

$$\iiint_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z+h)e^{-i(\alpha_1 x+\alpha_2 y+\alpha_3 z)}dx\,dy\,dz = e^{i\alpha_3 h} \qquad (2.37)$$

and the left-hand side transforms according to equations (2.31) and (2.36). Thus, one gets

$$\begin{bmatrix} \chi k_0^2-\alpha_2^2-\alpha_3^2 & \alpha_1(\alpha_2+i\kappa\alpha_3) & \alpha_1(-i\kappa\alpha_2+\alpha_3) \\ \alpha_1(\alpha_2-i\kappa\alpha_3) & \chi k_0^2-\alpha_1^2-\frac{\chi}{\zeta}\alpha_3^2 & \frac{\chi}{\zeta}\alpha_2\alpha_3+i\kappa\alpha_1^2 \\ \alpha_1(i\kappa\alpha_2+\alpha_3) & \frac{\chi}{\zeta}\alpha_2\alpha_3-i\kappa\alpha_1^2 & \chi k_0^2-\alpha_1^2-\frac{\chi}{\zeta}\alpha_2^2 \end{bmatrix} \begin{bmatrix} \tilde{\tilde{\tilde{H}}}_{x_1} \\ \tilde{\tilde{\tilde{H}}}_{y_1} \\ \tilde{\tilde{\tilde{H}}}_{z_1} \end{bmatrix} = \frac{-i\omega\epsilon_0\chi m}{(2\pi)^{3/2}} \begin{bmatrix} e^{i\alpha_3 h} \\ 0 \\ 0 \end{bmatrix} \qquad (2.38)$$

The above system of algebraic equations can be solved using Cramer's rule. The determinant of the coeficients of the square matrix on the left-hand side can be found to be

$$\Delta = \frac{\chi^2 k_0^2}{\zeta}\Big\{\alpha_3^4 - \Big[(\chi+\zeta)k_0^2 - \frac{\chi+\zeta(1-\kappa^2)}{\chi}\alpha_1^2 - 2\alpha_2^2\Big]\alpha_3^2$$
$$+\Big[\zeta\chi k_0^4 + \frac{\zeta(1-\kappa^2)}{\chi}\alpha_1^4 + \alpha_2^4 + \frac{\chi+\zeta(1-\kappa^2)}{\chi}\alpha_1^2\alpha_2^2 - 2\zeta k_0^2\alpha_1^2 - (\zeta+\chi)k_0^2\alpha_2^2\Big]\Big\}. \tag{2.39}$$

We recognize the above expression as a fourth order equation in α_3. It can be readily put in the form

$$\Delta = \frac{\chi^2 k_0^2}{\zeta}(\alpha_3^2 - s_1^2)(\alpha_3^2 - s_2^2) \tag{2.40}$$

where s_1^2 and s_2^2 are given by

$$s_{1,2}^2 = \frac{1}{2}\Big\{(\chi+\zeta)k_0^2 - \frac{\chi+\zeta(1-\kappa^2)}{\chi}\alpha_1^2 - 2\alpha_2^2 \pm \sqrt{\Big[(\chi-\zeta)k_0^2 - \frac{\chi-\zeta(1-\kappa^2)}{\chi}\alpha_1^2\Big]^2 + 4\zeta\kappa^2 k_0^2\alpha_1^2}\Big\} \tag{2.41}$$

The result of equation (2.41) is not surprising. It is typical of what could be expected from analysis of a birefringent medium like magnetoplasma. We found two characteristic modes of propagation, s_1 and s_2, with which terms "ordinary" and "extraordinary" waves are often associated. We note, moreover, that the determinant of equation (2.39) contains all possible information about the modes of propagation within the magnetoplasma. These modes are given by the zeroes of Δ, which are actually the poles of the field components. We note, for instance, that if $\alpha_2 = \alpha_3 = 0$ in equation (2.39), which is equivalent to saying that the wave does vary along the y and z coordinates, we obtain for the characteristic modes

$$(\alpha_1)_{1,2} = k_0\sqrt{\epsilon \pm \eta} \tag{2.42}$$

which can be immediately recognized as the plane wave propagation constants in the direction of the steady magnetic field (24, p. 297). Alternately, if we set $\alpha_1 = \alpha_3 = 0$ and solve for α_2 we obtain

$$(\alpha_2)_1 = k_o\sqrt{\chi}$$
$$(\alpha_2)_2 = k_o\sqrt{\zeta} \tag{2.43}$$

which again can be recognized as plane wave propagation constants transverse to the steady magnetic field (24, p. 297.) Setting $\alpha_1 = \alpha_2 = 0$ and solving for α_3 gives the same results as in (2.43) as it should.

We now proceed with formally solving the system of algebraic equations (2.38). Using Cramer's rule and the results of (2.40) and (2.41), we obtain

$$\begin{aligned}
\tilde{\tilde{H}}_{x_1}^{(P)} &= \frac{-i\omega\epsilon_o m}{(2\pi)^{3/2}\chi k_o^2}\cdot\frac{\Phi_1(\alpha_1,\alpha_2,\alpha_3)e^{i\alpha_3 h}}{(\alpha_3^2-s_1^2)(\alpha_3^2-s_2^2)}\\
\tilde{\tilde{H}}_{y_1}^{(P)} &= \frac{i\omega\epsilon_o m}{(2\pi)^{3/2}\chi k_o^2}\cdot\frac{\alpha_1[\alpha_2\Phi_2(\alpha_1,\alpha_2,\alpha_3)-i\kappa\chi\zeta k_o^2\alpha_3]e^{i\alpha_3 h}}{(\alpha_3^2-s_1^2)(\alpha_3^2-s_2^2)}\\
\tilde{\tilde{H}}_{z_1}^{(P)} &= \frac{i\omega\epsilon_o m}{(2\pi)^{3/2}\chi k_o^2}\cdot\frac{\alpha_1[\alpha_3\Phi_2(\alpha_1,\alpha_2,\alpha_3)+i\kappa\chi\zeta k_o^2\alpha_2]e^{i\alpha_3 h}}{(\alpha_3^2-s_1^2)(\alpha_3^2-s_2^2)}
\end{aligned} \tag{2.44}$$

where

$$\begin{aligned}
\Phi_1(\alpha_1,\alpha_2,\alpha_3) &= (\chi k_o^2-\alpha_1^2)[\zeta(\chi k_o^2-\alpha_1^2)-\chi(\alpha_2^2+\alpha_3^2)]-\kappa^2\zeta\alpha_1^4\\
\Phi_2(\alpha_1,\alpha_2,\alpha_3) &= \zeta\chi k_o^2-\zeta(1-\kappa^2)\alpha_1^2-\chi(\alpha_2^2+\alpha_3^2)
\end{aligned} \tag{2.45}$$

We are now ready to invert the field transforms with respect to the transform variable according to the prescription

$$\tilde{H}^{(P)}(\alpha_1,\alpha_2,z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\tilde{\tilde{H}}^{(P)}(\alpha_1,\alpha_2,\alpha_3)\, e^{i\alpha_3 z}\, d\alpha_3 \, . \tag{2.46}$$

To this end we consider the following integral

$$I = \int_{-\infty}^{\infty}\frac{\Phi\,(\alpha_1,\alpha_2,\alpha_3)\; e^{i\alpha_3(z+h)}}{(\alpha_3^2-s_1^2)\;(\alpha_3^2-s_2^2)}\, d\alpha_3 \tag{2.47}$$

where now we assume α_3 to be complex. The integrand has two pairs of poles in the complex α_3-plane which can be either real or complex. If they are

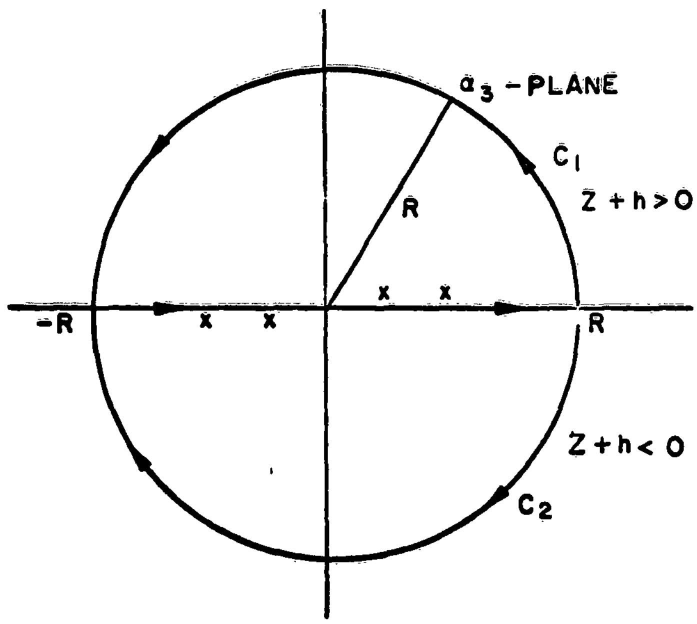

Figure 2.3 - The Complex α_3-Plane

complex we assume that

$$\begin{aligned} \mathrm{Im}\,\{ s_1 \} &\geqslant 0 \\ \mathrm{Im}\,\{ s_2 \} &\geqslant 0 \;. \end{aligned} \tag{2.48}$$

Then for $z + h > 0$ we close the contour in the upper half plane as shown in Figure 2.4 and write

$$\begin{aligned} \int_{C+C_1} \frac{\Phi(\alpha_3) e^{i\alpha_3(z+h)}}{(\alpha_3^2 - s_1^2)(\alpha_3^2 - s_2^2)}\, d\alpha_3 &= 2\pi i \sum \mathrm{Res.} \\ &= \int_{-R}^{R} \frac{\Phi(\alpha_3) e^{i\alpha_3(z+h)}}{(\alpha_3^2 - s_1^2)(\alpha_3^2 - s_2^2)}\, d\alpha_3 + \int_{C_1} \frac{\Phi(\alpha_3) e^{i\alpha_3(z+h)}}{(\alpha_3^2 - s_1^2)(\alpha_3^2 - s_2^2)}\, d\alpha_3 \end{aligned} \tag{2.49}$$

On C_1 we set $\alpha_3 = Re^{i\theta}$, then for $R \longrightarrow \infty$ we obtain for the second integral on the right

$$\int_{C_1} \xrightarrow[R \to \infty]{} i \int_0^{\pi} \frac{\Phi(Re^{i\theta}) e^{i(z+h)R\cos\theta} e^{-(z+h)R\sin\theta}}{R^3 e^{i3\theta}}\, d\theta \xrightarrow[R \to \infty]{} 0 \tag{2.50}$$

since the order of the numerator is never larger than three (see (2.44) and (2.45).) Thus

$$I = \int_{-\infty}^{\infty} \frac{\Phi(\alpha_3) e^{i\alpha_3(z+h)}}{(\alpha_3^2 - s_1^2)(\alpha_3^2 - s_2^2)}\, d\alpha_3 = 2\pi i \sum \mathrm{Res.} \tag{2.51}$$

Now

$$2\pi i \sum \mathrm{Res} = \pi i \left\{ \frac{\Phi(s_1) e^{is_1(z+h)}}{s_1(s_1^2 - s_2^2)} - \frac{\Phi(s_2) e^{is_2(z+h)}}{s_2(s_1^2 - s_2^2)} \right\} \tag{2.52}$$

and the integral in (2.51) becomes

$$I = \pi i \left\{ \frac{\Phi(s_1) e^{is_1(z+h)}}{s_1(s_1^2 - s_2^2)} - \frac{\Phi(s_2) e^{is_2(z+h)}}{s_2(s_1^2 - s_2^2)} \right\} \tag{2.53}$$

For $z + h < 0$ we close the contour in the lower half plane and by an analogous argument obtain

$$I = \pi i \left\{ \frac{\Phi(-s_1)e^{-is_1(z+h)}}{s_1(s_1^2-s_2^2)} - \frac{\Phi(-s_2)e^{-is_2(z+h)}}{s_2(s_1^2-s_2^2)} \right\} \tag{2.54}$$

Then the transforms of the magnetic field components when inverted with respect to the α_3 transform variable, become

$$\tilde{\tilde{H}}_{x_1}^{(p)} = \frac{\omega\epsilon_0 m}{4\pi\chi k_0^2} \left\{ \frac{\Phi_1(\alpha_1,\alpha_2,s_1)e^{is_1|z+h|}}{s_1(s_1^2-s_2^2)} - \frac{\Phi_1(\alpha_1,\alpha_2,s_2)e^{is_2|z+h|}}{s_2(s_1^2-s_2^2)} \right\} \tag{2.55a}$$

$$\tilde{\tilde{H}}_{y_1}^{(p)} \underset{z+h\gtrless 0}{=} \frac{-\omega\epsilon_0 m\alpha_1}{4\pi\chi k_0^2} \left\{ \frac{\alpha_2\Phi_2(\alpha_1,\alpha_2,s_1) \mp i\kappa\chi\zeta k_0^2 s_1}{s_1(s_1^2-s_2^2)} e^{is_1|z+h|} - \frac{\alpha_2\Phi_2(\alpha_1,\alpha_2,s_2) \mp i\kappa\chi\zeta k_0^2 s_2}{s_2(s_1^2-s_2^2)} e^{is_2|z+h|} \right\} \tag{2.55b}$$

$$\tilde{\tilde{H}}_{z_1}^{(p)} \underset{z+h\gtrless 0}{=} \frac{-\omega\epsilon_0 m\alpha_1}{4\pi\chi k_0^2} \left\{ \frac{\pm s_1\Phi_2(\alpha_1,\alpha_2,s_1) + i\kappa\chi\zeta k_0^2\alpha_2}{s_1(s_1^2-s_2^2)} e^{is_1|z+h|} - \frac{\pm s_2\Phi_2(\alpha_1,\alpha_2,s_2) + i\kappa\chi\zeta k_0^2\alpha_2}{s_2(s_1^2-s_2^2)} e^{is_2|z+h|} \right\}. \tag{2.55c}$$

Finally, we invert with respect to α_1 and α_2 to obtain the desired representations

$$H_{x_1}^{(p)} = \frac{\omega\epsilon_0 m}{8\pi^2\chi k_0^2} \iint_{-\infty}^{\infty} \left\{ \frac{\Phi_1(s_1)e^{is_1|z+h|}}{s_1(s_1^2-s_2^2)} - \frac{\Phi_1(s_2)e^{is_2|z+h|}}{s_2(s_1^2-s_2^2)} \right\} e^{i(\alpha_1 x+\alpha_2 y)} d\alpha_1 d\alpha_2 \tag{2.56a}$$

$$H_{y_1}^{(p)} \underset{z+h\gtrless 0}{=} \frac{i\omega\epsilon_0 m}{8\pi^2\chi k_0^2} \partial_x \iint_{-\infty}^{\infty} \left\{ \frac{\alpha_2\Phi_2(s_1) \mp i\kappa\chi\zeta k_0^2 s_1}{s_1(s_1^2-s_2^2)} e^{is_1|z+h|} - \frac{\alpha_2\Phi_2(s_2) \mp i\kappa\chi\zeta k_0^2 s_2}{s_2(s_1^2-s_2^2)} e^{is_2|z+h|} \right\} e^{i(\alpha_1 x+\alpha_2 y)} d\alpha_1 d\alpha_2 \tag{2.56b}$$

$$H_{z_1}^{(p)} \underset{z+h \gtrless 0}{=} \frac{i\omega\epsilon_0 m}{8\pi^2 \chi k_0^2} \partial_x \iint_{-\infty}^{\infty} \left\{ \frac{\pm s_1 \Phi_2(s_1) + i\kappa\chi\zeta k_0^2 \alpha_2}{s_1(s_1^2 - s_2^2)} e^{is_1|z+h|} - \frac{\pm s_2 \Phi_2(s_2) + i\kappa\chi\zeta k_0^2 \alpha_2}{s_2(s_1^2 - s_2^2)} e^{is_2|z+h|} \right\} e^{i(\alpha_1 x + \alpha_2 y)} d\alpha_1 d\alpha_2 \tag{2.56c}$$

<u>2.3b The complementary field in the plasma</u>--In the previous section we obtained the "particular integrals" of the system of equations (2.33) which represent the primary excitation due to the source. To satisfy the boundary conditions of the problem, we shall need an appropriate complementary solution of the homogeneous system of (2.33). From the theory of linear differential equations (10, p. 145) it follows that the magnetic field components satisfy the equation

$$(\alpha_3^2 - s_1^2)(\alpha_3^2 - s_2^2)\, \vec{\tilde{\tilde{H}}}^{(c)} = 0 \tag{2.57}$$

where α_3 in this equation is considered as the differential operator with respect to the z- coordinate. The solutions to (2.57) can be written down immediately

$$\begin{bmatrix} \tilde{\tilde{H}}_{x_1}^{(c)} \\ \tilde{\tilde{H}}_{y_1}^{(c)} \\ \tilde{\tilde{H}}_{z_1}^{(c)} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} \begin{bmatrix} e^{-is_1 z} \\ e^{-is_2 z} \end{bmatrix} \tag{2.58}$$

where we discarded solutions with positive exponentials since we can have only waves going away from the interface upon reflection. The coefficients C_{ij} are not all independent. For since $\vec{\tilde{\tilde{H}}}$ has to satisfy the given system, the coefficients must be related by

$$\begin{bmatrix} F_{11}(-s_1) & F_{12}(-s_1) & F_{13}(-s_1) \\ F_{21}(-s_1) & F_{22}(-s_1) & F_{23}(-s_1) \\ F_{31}(-s_1) & F_{32}(-s_1) & F_{33}(-s_1) \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix} = 0 \tag{2.59}$$

and

$$\begin{bmatrix} F_{11}(-s_2) & F_{12}(-s_2) & F_{13}(-s_2) \\ F_{21}(-s_2) & F_{22}(-s_2) & F_{23}(-s_2) \\ F_{31}(-s_2) & F_{32}(-s_2) & F_{33}(-s_2) \end{bmatrix} \begin{bmatrix} C_{12} \\ C_{22} \\ C_{32} \end{bmatrix} = 0 \qquad (2.60)$$

where the square matrix elements $F_{ij}(s_{1,2})$ correspond to the matrix of the original system (2.38) with $\alpha_3 = -s_1$ or $\alpha_3 = -s_2$. Now we express all coefficients C_{ij} in terms of C_{11} and C_{12}. The procedure is straight-forward and we obtain as the results

$$C_{21} = -\alpha_1 \cdot \frac{\alpha_2 \Phi_2(s_1) + i\kappa\chi\zeta k_0^2 s_1}{\Phi_1(s_1)} C_{11}$$
$$C_{31} = -\alpha_1 \cdot \frac{-s_1 \Phi_2(s_1) + i\kappa\chi\zeta k_0^2 \alpha_2}{\Phi_1(s_1)} C_{11} \qquad (2.61)$$

and

$$C_{22} = -\alpha_1 \cdot \frac{\alpha_2 \Phi_2(s_2) + i\kappa\chi\zeta k_0^2 s_2}{\Phi_1(s_2)} C_{12}$$
$$C_{32} = -\alpha_1 \cdot \frac{-s_2 \Phi_2(s_2) + i\kappa\chi\zeta k_0^2 \alpha_2}{\Phi_1(s_2)} C_{12} \qquad (2.62)$$

For convenience in what follows we shall normalize the coefficients C_{11} and C_{12} as follows:

$$C_{11} = \frac{\omega \epsilon_0 m}{4\pi \chi k_0^2} A_1$$
$$C_{12} = \frac{\omega \epsilon_0 m}{4\pi \chi k_0^2} A_2 \qquad (2.63)$$

Now, inverting with respect to α_1 and α_2, we obtain the representation of the complementary field in the plasma

$$H_{x1}^{(c)} = \frac{\omega\epsilon_0 m}{8\pi^2 \chi k_0^2} \iint_{-\infty}^{\infty} \left\{ A_1 e^{-is_1 z} + A_2 e^{-is_2 z} \right\} e^{i(\alpha_1 x + \alpha_2 y)} d\alpha_1 d\alpha_2 \qquad (2.64a)$$

$$H_{y1}^{(c)} = \frac{i\omega\epsilon_0 m}{8\pi^2 \chi k_0^2} \partial_x \iint_{-\infty}^{\infty} \left\{ \frac{\alpha_2 \Phi_2(s_1) + i\kappa\chi\zeta k_0^2 s_1}{\Phi_1(s_1)} A_1 e^{-is_1 z} + \frac{\alpha_2 \Phi_2(s_2) + i\kappa\chi\zeta k_0^2 s_2}{\Phi_1(s_2)} A_2 e^{-is_2 z} \right\} e^{i(\alpha_1 x + \alpha_2 y)} d\alpha_1 d\alpha_2 \qquad (2.64b)$$

$$H_{z1}^{(c)} = \frac{i\omega\epsilon_0 m}{8\pi^2\chi k_0^2}\partial_x \iint_{-\infty}^{\infty}\left\{\frac{-s_1\Phi_2(s_1)+i\kappa\chi\zeta k_0^2\alpha_2}{\Phi_1(s_1)}A_1 e^{-is_1 z} + \frac{-s_2\Phi_2(s_2)+i\kappa\chi\zeta k_0^2\alpha_2}{\Phi_1(s_2)}A_2 e^{-is_2 z}\right\} e^{i(\alpha_1 x+\alpha_2 y)}d\alpha_1 d\alpha_2 \tag{2.64c}$$

As a partial check on the above results we investigate whether $\vec{\nabla}\cdot\vec{H}^{(c)} = 0$ as it should. Indeed, we find that the above is true, providing

$$\Phi_1(s_{1,2}) = (\alpha_2^2 + s_{1,2}^2)\,\Phi_2(s_{1,2}) \tag{2.65}$$

which follow from (2.45) and (2.41). In this connection we note the following useful and not too obvious relationships

$$\chi(s_1^2 + s_2^2) = \chi(\chi k_0^2 - \alpha_1^2 - \alpha_2^2) + \zeta\chi k_0^2 - \zeta(1-\kappa^2)\alpha_1^2 - \chi\alpha_2^2 \tag{2.66a}$$

$$\chi s_{1,2}^2 = \Phi_2(s_{1,2}) + \chi(\chi k_0^2 - \alpha_1^2 - \alpha_2^2) \tag{2.66b}$$

$$\chi(s_1^2 - s_2^2) = \Phi_2(s_2) - \Phi_2(s_1) \tag{2.66c}$$

$$\Phi_2(s_{1,2}) = \chi(\alpha_1^2 + \alpha_2^2 + s_{1,2}^2 - \chi k_0^2) \tag{2.66d}$$

<u>2.3c The field in the air</u>—As we noted in section (2.2c) the fields in the air are derivable from the Hertzian vector potential of the magnetic type satisfying the homogeneous vector wave equation

$$(\nabla^2 + k_0^2)\vec{\Pi}_0 = 0 \quad .$$

The form of solution immediately suggests itself from examination of the complementary field in the plasma. We write

$$\Pi_{x0} = \frac{\omega\epsilon_0 m}{8\pi^2\chi k_0^4}\iint_{-\infty}^{\infty} B_1 e^{is_0 z}\, e^{i(\alpha_1 x+\alpha_2 y)}\,d\alpha_1 d\alpha_2 \tag{2.67a}$$

$$\Pi_{z0} = \frac{-\omega\epsilon_0 m}{8\pi^2\chi k_0^4}\iint_{-\infty}^{\infty} B_2\, e^{i(\alpha_1 x+\alpha_2 y+s_0 z)}\,d\alpha_1 d\alpha_2 \tag{2.67b}$$

where

$$s_0 = \sqrt{k_0^2 - \alpha_1^2 - \alpha_2^2}$$
$$\mathrm{Im}\,\{s_0\} \geq 0 \tag{2.68}$$

Here we assumed that the vector $\vec{\Pi}$ has only two non-vanishing components which are sufficient to derive all of the electric and magnetic field components and to satisfy the boundary conditions.

2.4 THE BOUNDARY CONDITIONS

In the preceding sections we have found field components in the plasma and air-regions that are solutions to Maxwell's equations. Moreover, in solving Maxwell's equations, we have chosen such solutions for field representation that have proper behavior at infinity by requiring that the imaginary part of the pertinent exponents be non-negative. In addition, the field components contain certain, thus far, undetermined coefficients which upon imposition of the boundary will be determined and thus render the solution unique.

The boundary conditions to be satisfied by the Cartesian components of the field vectors require continuity of the tangential components of the electric and magnetic fields at the interface $z = 0$. This implies the following:

$$\begin{aligned} H_{x0} &= H_{x1} \\ H_{y0} &= H_{y1} \\ E_{x0} &= E_{x1} \\ E_{y0} &= E_{y1} \end{aligned} \tag{2.69}$$

We rewrite the above equations using the Hertzian vector representation for the fields in the air

$$
\begin{aligned}
&k_o^2 \Pi_{xo} + \partial_x(\partial_x \Pi_{xo} + \partial_z \Pi_{zo}) = H_{x1} \\
&\partial_y(\partial_x \Pi_{xo} + \partial_z \Pi_{zo}) = H_{y1} \\
&\zeta k_o^2 \partial_y \Pi_{zo} = \partial_y H_{z1} - \partial_z H_{y1} \\
&\chi k_o^2(\partial_z \Pi_{xo} - \partial_x \Pi_{zo}) = (i\kappa \partial_y + \partial_z) H_{x1} - i\kappa \partial_x H_{y1} - \partial_x H_{z1}
\end{aligned}
\tag{2.70}
$$

The above system of equations can be somewhat simplified. One can show that

$$
\begin{aligned}
k_o^2 \check{\Pi}_{xo} &= \check{H}_{x1} - \frac{\alpha_1}{\alpha_2} \check{H}_{y1} \\
k_o^2 \check{\Pi}_{zo} &= -\frac{\alpha_1}{s_o} \check{H}_{x1} - \frac{k_o^2 - \alpha_1^2}{\alpha_2 s_o} \check{H}_{y1} \\
k_o^2 \check{\Pi}_{zo} &= \frac{1}{\zeta \alpha_2} \frac{\partial \check{H}_{y1}}{\partial z} + \frac{1}{\zeta} \check{H}_{z1} \\
k_o^2 \check{\Pi}_{xo} &= \frac{-i}{\chi s_o} \left(\frac{\partial}{\partial z} - \kappa \alpha_2\right) \check{H}_{x1} + \frac{i\alpha_1}{s_o}\left(\frac{1}{\alpha_2 \zeta} \frac{\partial}{\partial z} - \frac{\kappa}{\chi}\right) \check{H}_{y1} + \frac{\alpha_1(\chi - \zeta)}{\zeta \chi s_o} \check{H}_{z1}
\end{aligned}
\tag{2.71}
$$

where we have used $\vee$ to denote the field expressions with the integral signs removed. Elimination of $\check{\Pi}_{xo}$ and $\check{\Pi}_{zo}$ gives

$$
\begin{aligned}
0 &= \alpha_2(\partial_z - i\chi s_o - \kappa\alpha_2)\check{H}_{x1} - \alpha_1\left(\frac{\chi}{\zeta}\partial_z - i\chi s_o - \kappa\alpha_2\right)\check{H}_{y1} + i\alpha_1\alpha_2\left(\frac{\chi}{\zeta} - 1\right)\check{H}_{z1} \\
0 &= i\alpha_1\alpha_2\zeta \check{H}_{x1} + \left[-s_o\partial_z + i\zeta(k_o^2 - \alpha_1^2)\right]\check{H}_{y1} + i\alpha_2 s_o \check{H}_{z1}
\end{aligned}
\tag{2.72}
$$

Upon performing the necessary algebraic operations, one obtains two simultaneous equations in two unknowns, A_1 and A_2. We write the results in a symbolic form

$$
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}
\tag{2.73}
$$

where the various coefficients are given by

$$a_{11} = \alpha_2(s_1 + \chi s_0 - i\kappa\alpha_2) + \alpha_1^2\left(\frac{\chi}{\zeta}s_1 + \chi s_0 - i\kappa\alpha_2\right)\frac{[\alpha_2\Phi_2(s_1) + i\kappa\chi\zeta k_0^2 s_1]}{\Phi_1(s_1)} + \alpha_1^2\alpha_2(\chi-\zeta)\frac{[-s_1\Phi_2(s_1) + i\kappa\chi\zeta k_0^2\alpha_2]}{\zeta\Phi_1(s_1)} \tag{2.74a}$$

$$a_{12} = \alpha_2(s_2 + \chi s_0 - i\kappa\alpha_2) + \alpha_1^2\left(\frac{\chi}{\zeta}s_2 + \chi s_0 - i\kappa\alpha_2\right)\frac{[\alpha_2\Phi_2(s_2) + i\kappa\chi\zeta k_0^2 s_2]}{\Phi_1(s_2)} + \alpha_1^2\alpha_2(\chi-\zeta)\frac{[-s_2\Phi_2(s_2) + i\kappa\chi\zeta k_0^2\alpha_2]}{\zeta\Phi_1(s_2)} \tag{2.74b}$$

$$\begin{aligned} b_1 = {} & \Big[\alpha_2(s_1 - \chi s_0 + i\kappa\alpha_2)\Phi_1(s_1) + \alpha_1^2\left(\frac{\chi}{\zeta}s_1 - \chi s_0 + i\kappa\alpha_2\right)\left(\alpha_2\Phi_2(s_1) - i\kappa\chi\zeta k_0^2 s_1\right) \\ & \quad - \alpha_1^2\alpha_2\left(\frac{\chi}{\zeta}-1\right)\left(s_1\Phi_2(s_1) + i\kappa\chi\zeta k_0^2\alpha_2\right)\Big]\frac{e^{is_1h}}{s_1(s_1^2 - s_2^2)} \\ & - \Big[\alpha_2(s_2 - \chi s_0 + i\kappa\alpha_2)\Phi_1(s_2) + \alpha_1^2\left(\frac{\chi}{\zeta}s_2 - \chi s_0 + i\kappa\alpha_2\right)\left(\alpha_2\Phi_2(s_2) - i\kappa\chi\zeta k_0^2 s_2\right) \\ & \quad - \alpha_1^2\alpha_2\left(\frac{\chi}{\zeta}-1\right)\left(s_2\Phi_2(s_2) + i\kappa\chi\zeta k_0^2\alpha_2\right)\Big]\frac{e^{is_2h}}{s_2(s_1^2 - s_2^2)} \end{aligned} \tag{2.74c}$$

and

$$a_{21} = \frac{\alpha_2[\Phi_1(s_1) - (s_0^2+\alpha_2^2)\Phi_2(s_1)] - i\kappa\chi k_0^2[\zeta s_1(s_0^2+\alpha_2^2) + s_0(s_1^2+\alpha_2^2)]}{\Phi_1(s_1)} \tag{2.75a}$$

$$a_{22} = \frac{\alpha_2[\Phi_1(s_2) - (s_0^2+\alpha_2^2)\Phi_2(s_2)] - i\kappa\chi k_0^2[\zeta s_2(s_0^2+\alpha_2^2) + s_0(s_2^2+\alpha_2^2)]}{\Phi_1(s_2)} \tag{2.75b}$$

$$\begin{aligned} b_2 = {} & -\left\{\alpha_2[\Phi_1(s_1) - (s_0^2+\alpha_2^2)\Phi_2(s_1)] + i\kappa\chi k_0^2[\zeta s_1(s_0^2+\alpha_2^2) - s_0(s_1^2+\alpha_2^2)]\right\}\frac{e^{is_1h}}{s_1(s_1^2 - s_2^2)} \\ & + \left\{\alpha_2[\Phi_1(s_2) - (s_0^2+\alpha_2^2)\Phi_2(s_2)] + i\kappa\chi k_0^2[\zeta s_2(s_0^2+\alpha_2^2) - s_0(s_2^2+\alpha_2^2)]\right\}\frac{e^{is_2h}}{s_2(s_1^2 - s_2^2)} \end{aligned} \tag{2.75c}$$

Once the coefficients A_1 and A_2 are found, the two remaining coefficients B_1 and B_2 can be determined from equations (2.71.) Thus, we find

$$\alpha_2 B_1 = \left\{\alpha_2\left[\Phi_1(s_1)+\alpha_1^2\Phi_2(s_1)\right]+i\kappa\chi\zeta k_0^2 s_1\alpha_1^2\right\}\frac{A_1}{\Phi_1(s_1)}$$
$$+\left\{\alpha_2\left[\Phi_1(s_2)+\alpha_1^2\Phi_2(s_2)\right]+i\kappa\chi\zeta k_0^2 s_2\alpha_1^2\right\}\frac{A_2}{\Phi_1(s_2)}$$
$$+\left\{\alpha_2\left[\Phi_1(s_1)+\alpha_1^2\Phi_2(s_1)\right]-i\kappa\chi\zeta k_0^2 s_1\alpha_1^2\right\}\frac{e^{is_1h}}{s_1(s_1^2-s_2^2)} \tag{2.76}$$
$$-\left\{\alpha_2\left[\Phi_1(s_2)+\alpha_1^2\Phi_2(s_2)\right]-i\kappa\chi\zeta k_0^2 s_2\alpha_1^2\right\}\frac{e^{is_2h}}{s_2(s_1^2-s_2^2)}$$

and

$$\frac{\alpha_2 s_0}{\alpha_1}B_2 = \left\{\alpha_2\left[\Phi_1(s_1)-(s_0^2+\alpha_2^2)\Phi_2(s_1)\right]-i\kappa\chi\zeta k_0^2 s_1(s_0^2+\alpha_2^2)\right\}\frac{A_1}{\Phi_1(s_1)}$$
$$+\left\{\alpha_2\left[\Phi_1(s_2)-(s_0^2+\alpha_2^2)\Phi_2(s_2)\right]-i\kappa\chi\zeta k_0^2 s_2(s_0^2+\alpha_2^2)\right\}\frac{A_2}{\Phi_1(s_2)}$$
$$+\left\{\alpha_2\left[\Phi_1(s_1)-(s_0^2+\alpha_2^2)\Phi_2(s_1)\right]+i\kappa\chi\zeta k_0^2 s_1(s_0^2+\alpha_2^2)\right\}\frac{e^{is_1h}}{s_1(s_1^2-s_2^2)} \tag{2.77}$$
$$-\left\{\alpha_2\left[\Phi_1(s_2)-(s_0^2+\alpha_2^2)\Phi_2(s_2)\right]+i\kappa\chi\zeta k_0^2 s_2(s_0^2+\alpha_2^2)\right\}\frac{e^{is_2h}}{s_2(s_1^2-s_2^2)}$$

The explicit determination of the four boundary coefficients A_1, A_2, B_1, and B_2 is now a straight-forward although a tedious matter. The results would necessarily be lengthy and probably not very useful. We, therefore, forego their evaluation in this form. In the next chapter, we shall introduce a high frequency approximation which will simplify the results a great deal without the loss of the basic ingredients of the problem.

2.5 CLOSURE

In the foregoing chapter we have rigorously formulated the problem of a horizontal magnetic dipole in a magnetoplasma with a separation boundary. We carried the analysis up to the point where the determination of the pertinent boundary coefficients remained to be a straight-forward (but not simple) algebraic process. Due to the algebraic complexity involved in explicit finding of the desired boundary coefficients it was concluded that their

final forms, if they could be found, would be of limited value in interpretation of their physical significance. Therefore, their explicit evaluation was postponed until the next chapter in which we anticipate the introduction of an approximation that would make the results algebraically manageable and their physical interpretation feasible.

CHAPTER 3

HIGH FREQUENCY APPROXIMATION FOR THE DIPOLE IN MAGNETOPLASMA

In the preceding chapter we have formulated the present boundary value problem by finding the integral expressions for the fields in the plasma and air-regions. These integral expressions contain certain boundary coefficients which are expressible in terms of the media properties, source strength and frequency. Even though in theory these coefficients are known, their usefulness in reducing the field integrals to form suitable for numerical calculations is of very limited value due to their complexity.

In this chapter we shall be concerned with finding appropriate approximations that would put the previously derived results in manageable form which could be interpreted physically.

3.1 THE NATURE OF THE APPROXIMATION

Any approximation that we may introduce to simplify the field components is necessarily contingent upon simplification of the propagation factors corresponding to the z-direction, i.e., s_1 and s_2 which are given by equation (2.41). Probably the most useful approximation will be one mentioned briefly in Section 2.2 and based on the assumption

$$\left| \frac{e B_{DC}}{\omega m} \right| \ll 1 \tag{3.1}$$

which can be brought about by either a weak steady magnetic field, H_{DC}, or by a sufficiently high wave frequency, ω. As a consequence, when the losses in the plasma are negligible, the plasma parameters can be written (see Section

2.2)

$$\begin{aligned}\zeta &= n^2\\ \epsilon &\sim n^2\\ \eta &\sim -\sigma(1-n^2)\\ \chi &\sim n^2\\ \kappa &\sim -\sigma\left(\frac{1-n^2}{n^2}\right)\end{aligned} \tag{3.2}$$

where "n" is the index of refraction of the plasma in absence of the steady magnetic field, $n = \sqrt{1-p^2}$. In what follows we shall require κ to be small. This will then exclude cases in which "n" is close to zero.

We note that, consistent with the approximations of equation (3.2), the permittivity tensor (2.8) and its inverse (2.9) become to the first order in κ

$$\overleftrightarrow{\epsilon} = n^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i\kappa \\ 0 & -i\kappa & 1 \end{bmatrix} \tag{3.3}$$

and

$$\overleftrightarrow{\epsilon}^{-1} = \frac{1}{n^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -i\kappa \\ 0 & i\kappa & 1 \end{bmatrix}. \tag{3.4}$$

3.1a Approximate forms of s_1 and s_2 — The approximations of equations (3.2) are now introduced into the equation (2.41) giving approximate forms for s_1 and s_2 as follows:

$$s_{1,2}^2 = s^2 + \frac{\kappa^2\alpha_1^2}{2} \pm \sqrt{\kappa^2 k_1^2\alpha_1^2 + \frac{\kappa^4\alpha_1^4}{4}} \tag{3.5a}$$

where

$$s^2 = k_1^2 - \alpha_1^2 - \alpha_2^2 \tag{3.5b}$$

$$k_1 = n\,k_0 \quad . \tag{3.5c}$$

If only first order terms in κ are kept, the above expression simplifies to

$$s_{1,2}^2 = s^2 \pm \kappa k_1 \alpha_1 \quad . \tag{3.6}$$

Finally, in regions where $|s^2| >> |\kappa k_1\alpha_1|$ one can further approximate by writing

$$s_{1,2} = s \pm \frac{\kappa k_1 \alpha_1}{2s} \quad . \tag{3.7}$$

In what followe we shall concl de that the last approximation is indeed the only useful one when calculating the amplitudes of the field components in the air. The usefulness of it is, however, reduced by the fact that it is not valid in the vicinity of $s = 0$ which corresponds to the angle of the total internal reflection in the plasma.

3.1b Approximate forms of Φ_1 and Φ_2 —Since both Φ_1 and Φ_2 contain only squares of s_1 and s_2, their approximate form need not be restricted by the requirement that $|s^2| >> |\kappa k_1\alpha_1|$. Thus, using (3.6) only, we rewrite (2.65) and (2.66)

$$\Phi_2(s_{1,2}) = \mp n^2 \kappa k_1 \alpha_1 \tag{3.8}$$

and

$$\Phi_1(s_{1,2}) = \mp n^2 \kappa k_1 \alpha_1 \, (s^2 + \alpha_2^2 \pm \kappa k_1 \alpha_1) \, . \tag{3.9}$$

We also note for future reference

$$\Phi_1(s_{1,2}) + \alpha_1^2 \Phi_2(s_{1,2}) = \mp n^2 \kappa k_1^2 \alpha_1 \, (k_1 \pm \kappa \alpha_1) \, . \tag{3.10}$$

3.2 APPROXIMATE FORMS OF THE BOUNDARY COEFFICIENTS

In the prece ding section we have derived certain approximations that are valid when the steady magnetic field is small enough or the wave frequency is high enough. Application of this approximation to the expression for the propagation factors s_1 and s_2 resulted in considerable simplification. Subsequently we found it necessary to introduce another simplification which resulted in limiting the range of validity of the approximation for the region where $|s^2| >> |\kappa k_1 \alpha_1|$.

In this section we shall apply this approximation to obtain simplified

expressions for the field components in the air.

We apply equations (3.6) and (3.8) through (3.10) to the matrix coefficients of equations (2.74) and (2.75) to obtain

$$\begin{aligned} a_{11} &= c - \kappa d \\ a_{12} &= c^* + \kappa d^* \\ a_{22} &= e - \kappa f \\ a_{21} &= e^* + \kappa f^* \end{aligned} \tag{3.11}$$

where

$$c = \frac{k_1(\alpha_2 k_1 - i\alpha_1 s)(s + n^2 s_0)}{s^2 + \alpha_2^2} \tag{3.12a}$$

$$e = \frac{k_1\{k_1(n^2-1)\alpha_1\alpha_2 - i[n^2 s(s_0^2+\alpha_2^2) + s_0(s^2+\alpha_2^2)]\}}{n^2\alpha_1(s^2+\alpha_2^2)} \tag{3.12b}$$

$$d = \frac{k_1 n^2\{\alpha_1\alpha_2[2\alpha_1^2 s s_0 + k_0^2(s^2-\alpha_2^2)] + ik_1[2k_0^2\alpha_2 s - \alpha_1^2 s_0(s^2-\alpha_2^2)]\}}{2s(s^2+\alpha_2^2)^2} \tag{3.12c}$$

$$f = \frac{k_1(s_0^2+\alpha_2^2)[2\alpha_1\alpha_2 s + ik_1(s^2-\alpha_2^2)]}{2s(s^2+\alpha_2^2)^2} \tag{3.12d}$$

Moreover,

$$b_1 = -\frac{n^2 k_1^2}{s}\left\{\alpha_2(s-n^2 s_0) + \frac{i\kappa}{2}\left[\alpha_1^2 + 2\alpha_2^2 + ih\alpha_1^2(s-n^2 s_0)\right]\right\}e^{ish} \tag{3.13a}$$

$$b_2 = \frac{k_1^2}{s^3}\left\{(n^2-1)\alpha_2 s^2 + \frac{i\kappa}{2}\left[s_0(s^2-\alpha_2^2) - ihs\left(n^2 s(s_0^2+\alpha_2^2) - s_0(s^2+\alpha_2^2)\right)\right]\right\}e^{ish} \tag{3.13b}$$

We also rewrite equations (2.76) and (2.77) in the same approximate form

$$\begin{aligned} B_1 = \frac{k_1}{\alpha_2(s^2+\alpha_2^2)^2}\Big\{&\alpha_2 k_1(s^2+\alpha_2^2)(A_1+A_2) - i\alpha_1 s(s^2+\alpha_2^2)(A_1-A_2) \\ &- \kappa\alpha_1^2\left[\alpha_1\alpha_2(A_1-A_2) - \frac{ik_1}{2s}(s^2-\alpha_2^2)(A_1+A_2)\right]\Big\} \\ &- \frac{k_1^2 n^2}{\alpha_2 s}\left(\alpha_2 - \frac{\kappa h\alpha_1^2}{2}\right)e^{ish} \end{aligned} \tag{3.14}$$

and

$$B_2 = \frac{1}{\alpha_2 s_0 (s^2+\alpha_2^2)^2}\Big\{(n^2-1)k_0^2\alpha_1\alpha_2(s^2+\alpha_2^2)(A_1+A_2) + ik_1 s(s^2+\alpha_2^2)(s_0^2+\alpha_2^2)(A_1-A_2)$$

$$+\kappa k_1\alpha_1(s_0^2+\alpha_2^2)\Big[\alpha_1\alpha_2(A_1-A_2) - \frac{ik_1}{2s}(s^2-\alpha_2^2)(A_1+A_2)\Big]\Big\}$$

$$-\frac{n^2\alpha_1}{s s_0\alpha_2}\Big[(n^2-1)k_0^2\alpha_2 + \frac{\kappa k_1^2 h}{2}(s_0^2+\alpha_2^2)\Big]e^{ish} \tag{3.15}$$

Next we evaluate $(A_1 + A_2)$ and $(A_1 - A_2)$ where A_1 and A_2 are given by equation (2.73). For the determinant of the square matrix we find

$$\Delta = \frac{-i2k_0^2 k_1\alpha_2}{\alpha_1 s(s^2+\alpha_2^2)^2}(s+s_0)\big[s(s^2+\alpha_2^2)(s+n^2 s_0) - i\kappa n^2\alpha_2(\alpha_1^2 s_0 + k_0^2 s)\big] \tag{3.16}$$

and

$$\frac{1}{\Delta} = \frac{i\alpha_1\big[s(s^2+\alpha_2^2)(s+n^2 s_0) + i\kappa n^2\alpha_2(\alpha_1^2 s_0 + k_0^2 s)\big]}{2k_0^2 k_1\alpha_2 s(s+s_0)(s+n^2 s_0)^2}\,. \tag{3.17}$$

Now, using Cramer's rule we find from equation (2.73)

$$A_1 = \frac{a_{22}b_1 - a_{12}b_2}{\Delta}$$
$$A_2 = \frac{-(a_{21}b_1 - a_{11}b_2)}{\Delta}\,. \tag{3.18}$$

Employing the notation of equation (3.11) one can show that

$$(A_1+A_2)\Delta = 2\big[i\,\mathrm{Im}(e) - \kappa\,\mathrm{Re}(f)\big]b_1 + 2\big[i\,\mathrm{Im}(c) - \kappa\,\mathrm{Re}(d)\big]b_2 \tag{3.19a}$$

and

$$(A_1-A_2)\Delta = 2\big[\mathrm{Re}(e) - i\kappa\,\mathrm{Im}(f)\big]b_1 - 2\big[\mathrm{Re}(c) - i\kappa\,\mathrm{Im}(d)\big]b_2 \tag{3.19b}$$

which amounts to

$$(A_1+A_2)\Delta = \frac{-i2k_1^3\alpha_2}{\alpha_1 s^2(s^2+\alpha_2^2)}\Big\{(n^2-1)s\big[(\alpha_1^2 s^2 - k_1^2\alpha_2^2) + s s_0(s^2+\alpha_2^2)\big]$$

$$+\frac{i\kappa n^2\alpha_2}{s^2+\alpha_2^2}\Big[-ihk_0^2(n^2-1)\alpha_1^2 s(s^2+\alpha_2^2) + k_0^2(n^2-1)\alpha_1^2\alpha_2^2 \tag{3.20a}$$

$$-k_0^2 s^2(s^2+\alpha_2^2) + \alpha_1^2 s^2(s_0^2+\alpha_2^2) - s s_0(s^2+\alpha_2^2)(k_0^2+\alpha_1^2)\Big]\Big\}\,.$$

and

$$(A_1 - A_2)\Delta = \frac{-k_1^4\alpha_2}{s^3(s^2+\alpha_2^2)}\Big\{4(n^2-1)\alpha_2 s^3$$

$$+\frac{i\kappa}{s^2+\alpha_2^2}\Big[-ih(n^2-1)s(s^2+\alpha_2^2)\big(k_1^2\alpha_2^2-\alpha_1^2 s^2+s s_0(s^2+\alpha_2^2)\big) \tag{3.20b}$$

$$+(s^2-\alpha_2^2)\big((n^2-1)(\alpha_1^2 s^2-k_1^2\alpha_2^2)+s s_0(n^2+1)(s^2+\alpha_2^2)\big)\Big]\Big\}.$$

Substituting the above results into equations (3.14) and (3.15) and also using equations (3.16) and (3.17) one obtains finally

$$B_1 = -n^2 k_1^2\left\{\frac{2}{s+n^2 s_0}+i\kappa\left[\frac{2\alpha_2}{(s+n^2 s_0)^2}+\frac{\alpha_1^2 s_0}{\alpha_2 s(s+s_0)(s+n^2 s_0)}+\frac{ih\alpha_1^2}{\alpha_2(s+n^2 s_0)}\right]\right\}e^{ihs} \tag{3.21a}$$

and

$$B_2 = -\alpha_1 k_1^2\Big\{\frac{2(n^2-1)}{(s+s_0)(s+n^2 s_0)} + i\kappa n^2\Big[\frac{2\alpha_2}{s(s+n^2 s_0)^2}-\frac{s_0^2+\alpha_2^2}{\alpha_2 s(s+s_0)(s+n^2 s_0)}$$

$$-ih\left(\frac{k_0^2}{\alpha_2 s(s+s_0)}-\frac{\alpha_1^2}{\alpha_2 s(s+n^2 s_0)}\right)\Big]\Big\}e^{ish}. \tag{3.21b}$$

We note that in the above boundary coefficients we have clearly separated the components that are dependent on the steady magnetic field (preceded by κ) from those that are not. Moreover, we note that the components of B_1 and B_2 which are independent of the steady magnetic field are the proper boundary coefficients for the analogous problem of the magnetic dipole in an isotropic half-space (2, p. 26–28).[+]

The components of B_1 and B_2 produced by the steady magnetic field display some heretofore not encountered denominators which, as we shall see later, are responsible for the unusual effects produced by the plasma's anisotropy.

[+] The reference quoted contains solution to an electric dipole. The comparison with our results can still be made by changing the appropriate transmission coefficients to the magnetic type.

3.3 THE COMPONENTS OF THE HERTZIAN VECTOR IN THE AIR

In the previous sections we found the approximate forms of the boundary coefficients that determine the amplitudes of the components of the Hertzian vector in the air. The pertinent expressions are necessarily long even in their simplest form. In this section we shall endeavor to arrange the results for the air-region in a suitable form for approximate evaluation of the necessary integrals.

<u>3.3a The Hertzian vector in Cartesian coordinates</u>—We substitute the results of equation (3.21) into the formulation of the Hertzian vector in equation (2.67) and obtain

$$\Pi_{x0} = \frac{-\omega\epsilon_0 n^2 m}{8\pi^2 k_0^2}\left\{ \iint_{-\infty}^{\infty} \frac{2}{s+n^2 s_0} e^{i(\alpha_1 x+\alpha_2 y+s_0 z+sh)}\, d\alpha_1\, d\alpha_2 \right.$$

$$+ iK \iint_{-\infty}^{\infty} \left[\frac{2\alpha_2}{(s+n^2 s_0)^2} + \frac{\alpha_1^2 s_0}{\alpha_2 s(s+s_0)(s+n^2 s_0)} + \frac{ih\alpha_1^2}{\alpha_2(s+n^2 s_0)} \right]$$

$$\left. \cdot\, e^{i(\alpha_1 x+\alpha_2 y+s_0 z+sh)}\, d\alpha_1\, d\alpha_2 \right\} \qquad (3.22a)$$

and

$$\Pi_{z0} = \frac{\omega\epsilon_0 n^2 m}{8\pi^2 k_0^2}\left\{ 2\left(\frac{n^2-1}{n^2}\right) \iint_{-\infty}^{\infty} \frac{\alpha_1}{(s+s_0)(s+n^2 s_0)} e^{i(\alpha_1 x+\alpha_2 y+s_0 z+sh)}\, d\alpha_1\, d\alpha_2 \right.$$

$$+ iK \iint_{-\infty}^{\infty} \left[\frac{2\alpha_1\alpha_2}{s(s+n^2 s_0)^2} - \frac{\alpha_1(s_0^2+\alpha_2^2)}{\alpha_2 s(s+s_0)(s+n^2 s_0)} \right.$$

$$\left. \left. - ih\left(\frac{\alpha_1 k_0^2}{\alpha_2 s(s+s_0)} - \frac{\alpha_1^3}{\alpha_2 s(s+n^2 s_0)} \right) \right] e^{i(\alpha_1 x+\alpha_2 y+s_0 z+sh)}\, d\alpha_1\, d\alpha_2 \right\}. \qquad (3.22b)$$

Using the identities

$$\frac{1}{s+s_0} = \frac{s-s_0}{k_0^2(n^2-1)} \qquad (3.23a)$$

and

$$\begin{aligned}
\alpha_1 &\Longleftrightarrow -i\partial_x \\
\alpha_2 &\Longleftrightarrow -i\partial_y \\
s_0 &\Longleftrightarrow -i\partial_z \\
s &\Longleftrightarrow -i\partial_h
\end{aligned} \tag{3.23b}$$

we can recast the above integrals in a simpler form as follows:

$$\begin{aligned}
\Pi_{x0} = \frac{-\omega\epsilon_0 n^2 m}{8\pi^2 k_0^2}\Bigg\{ &\iint_{-\infty}^{\infty} \frac{2}{s+n^2 s_0} e^{i(\alpha_1 x+\alpha_2 y+s_0 z+sh)}\, d\alpha_1 d\alpha_2 \\
&+\kappa\Bigg[2\partial_y \iint_{-\infty}^{\infty} \frac{1}{(s+n^2 s_0)^2} e^{i(\alpha_1 x+\alpha_2 y+s_0 z+sh)}\, d\alpha_1 d\alpha_2 \\
&\quad -\frac{i}{(1-n^2)k_0^2}\partial_x^2 \partial_z(\partial_h-\partial_z) \iint_{-\infty}^{\infty} \frac{1}{\alpha_2 s(s+n^2 s_0)} e^{i(\alpha_1 x+\alpha_2 y+s_0 z+sh)}\, d\alpha_1 d\alpha_2 \\
&\quad +h\partial_x^2 \iint_{-\infty}^{\infty} \frac{1}{\alpha_2(s+n^2 s_0)} e^{i(\alpha_1 x+\alpha_2 y+s_0 z+sh)}\, d\alpha_1 d\alpha_2 \Bigg]\Bigg\}
\end{aligned} \tag{3.24a}$$

and

$$\begin{aligned}
\Pi_{z0} = \frac{-i\omega\epsilon_0 n^2 m}{8\pi^2 k_0^2}\partial_x\Bigg\{ &-\frac{i2}{k_1^2}(\partial_h-\partial_z) \iint_{-\infty}^{\infty} \frac{1}{s+n^2 s_0} e^{i(\alpha_1 x+\alpha_2 y+s_0 z+sh)}\, d\alpha_1 d\alpha_2 \\
&+\kappa\Bigg[2\partial_y \iint_{-\infty}^{\infty} \frac{1}{s(s+n^2 s_0)^2} e^{i(\alpha_1 x+\alpha_2 y+s_0 z+sh)}\, d\alpha_1 d\alpha_2 \\
&\quad -\frac{1}{(1-n^2)k_0^2}(\partial_y^2+\partial_z^2)(\partial_h-\partial_z) \iint_{-\infty}^{\infty} \frac{1}{\alpha_2 s(s+n^2 s_0)} e^{i(\alpha_1 x+\alpha_2 y+s_0 z+sh)}\, d\alpha_1 d\alpha_2 \\
&\quad +h\Bigg(\frac{i}{1-n^2}(\partial_h-\partial_z) \iint_{-\infty}^{\infty} \frac{1}{\alpha_2 s} e^{i(\alpha_1 x+\alpha_2 y+s_0 z+sh)}\, d\alpha_1 d\alpha_2 \\
&\quad +\partial_x^2 \iint_{-\infty}^{\infty} \frac{1}{\alpha_2 s(s+n^2 s_0)} e^{i(\alpha_1 x+\alpha_2 y+s_0 z+sh)}\, d\alpha_1 d\alpha_2\Bigg)\Bigg]\Bigg\} .
\end{aligned} \tag{3.24b}$$

In the above expressions we have clearly separated the contributions of the steady magnetic field in the plasma by enclosing them in square brackets. The first term in each of the above Hertzian vector components can be recognized as the proper expression for the problem of a magnetic dipole in a dielectric with an index of refraction n.

3.3 b Definition of the fundamental integrals--To facilitate the approximate evaluation of the various integrals in equations (3.24a) and (3.24b) we shall define certain fundamental integrals from which all others could be derived by differentiation. We define for convenience

$$U_1 = \frac{1}{k_0 \pi} \iint_{-\infty}^{\infty} \frac{1}{s + n^2 s_0} e^{i(\alpha_1 x + \alpha_2 y + s_0 z + s h)} d\alpha_1 d\alpha_2 \tag{3.25a}$$

$$U_2 = \frac{k_0}{2\pi i} \iint_{-\infty}^{\infty} \frac{1}{s \alpha_2 (s + n^2 s_0)} e^{i(\alpha_1 x + \alpha_2 y + s_0 z + s h)} d\alpha_1 d\alpha_2 \tag{3.25b}$$

$$U_3 = \frac{k_0}{\pi} \iint_{-\infty}^{\infty} \frac{1}{s (s + n^2 s_0)} e^{i(\alpha_1 x + \alpha_2 y + s_0 z + s h)} d\alpha_1 d\alpha_2 \tag{3.25c}$$

The above integrals are not independent. Indeed, they are related as follows:

$$\begin{aligned} U_1 &= -\frac{2i}{k_0^2} \partial_y \partial_h U_2 \\ 2i\, \partial_y U_2 &= (\partial_h + n^2 \partial_z)\, U_3 \end{aligned} \tag{3.26}$$

In what follows, we shall avail ourselves of the above relation to check the differentiability of our results after integration is performed.

Employing the above definitions of U_1, U_2, and U_3 we can rewrite the Hertzian vector components in terms of them and obtain

$$\Pi_{x0} = \frac{-n^2 m k_0}{4\pi \omega \mu_0} \left\{ U_1 - \frac{i K}{k_0} \left[\partial_y \partial_h U_3 + i \partial_x^2 \left(\frac{1}{(1-n^2) k_0^2} \partial_z (\partial_h - \partial_z) + h \partial_h \right) U_3 \right] \right\} \tag{3.27a}$$

and

$$\Pi_{zo} = \frac{-n^2 m}{4\pi k_o \omega \mu_o} \partial_x \left\{ \frac{1}{n^2}(\partial_h - \partial_z) U_1 + i\kappa \left[\partial_y U_3 - \frac{i}{(1-n^2)k_o^2}(\partial_y^2 + \partial_z^2)(\partial_h - \partial_z) U_2 \right.\right.$$

$$\left.\left. + ih\left(\frac{1}{1-n^2}(\partial_h - \partial_z)(\partial_h + n^2\partial_z) + \partial_x^2\right) U_2 \right\} \tag{3.27b}$$

3.3c Transformation to cylindrical coordinates in configuration and transform spaces—The above integral representations are surface integrals over the entire $\alpha_1 - \alpha_2$ plane of a form that lends itself readily to a transformation to cylindrical coordinates in the configuration space as well as in the transformation space. Thus, we employ the transformation

$$x = \rho \cos\varphi$$
$$y = \rho \sin\varphi \tag{3.28}$$
$$\rho = \sqrt{x^2 + y^2}$$

for the space coordinates and

$$\alpha_1 = \lambda \cos\beta$$
$$\alpha_2 = \lambda \sin\beta \tag{3.29}$$
$$\lambda = \sqrt{\alpha_1^2 + \alpha_2^2}$$

for the transform variables. As the consequence of the above transformations, we note the following:

$$\alpha_1 x + \alpha_2 y = \lambda \rho \cos(\beta - \varphi)$$
$$s_o = \sqrt{k_o^2 - \lambda^2}$$
$$s = \sqrt{k_1^2 - \lambda^2} \tag{3.30}$$
$$d\alpha_1 \, d\alpha_2 = \lambda \, d\lambda \, d\beta \quad .$$

The integrals appearing in the previous sections are of two different types. The first one is

$$I_1 = \iint\limits_{-\infty}^{\infty} F_1(\alpha_1^2, \alpha_2^2) e^{i(\alpha_1 x + \alpha_2 y)} d\alpha_1 d\alpha_2$$

$$= \int_0^{\infty} F_1(\lambda^2)\lambda d\lambda \int_{-\pi}^{\pi} e^{i\lambda\rho\cos(\beta-\varphi)} d\beta \tag{3.31}$$

$$= 2\pi \int_0^{\infty} F_1(\lambda^2) J_0(\lambda\rho)\lambda d\lambda$$

where we used the well-known representation of Bessel functions (22, p. 273)

$$J_\nu(\lambda\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\lambda\rho\cos(\beta-\varphi) + i\nu(\beta-\varphi-\pi/2)} d\beta \,. \tag{3.32}$$

The second distinct integral is of the form

$$I_2 = \iint\limits_{-\infty}^{\infty} \frac{F_2(\alpha_1^2, \alpha_2^2)}{\alpha_2} e^{i(\alpha_1 x + \alpha_2 y)} d\alpha_1 d\alpha_2 \tag{3.33}$$

$$= \int_0^{\infty} F_2(\lambda^2) d\lambda \int_{-\pi}^{\pi} \frac{e^{i\lambda\rho\cos(\beta-\varphi)}}{\sin\beta} d\beta \,.$$

It is shown in Appendix A that

$$\int_{-\pi}^{\pi} \frac{e^{i\lambda\rho\cos(\beta-\varphi)}}{\sin\beta} d\beta = 4\pi i \sum_{\nu=0}^{\infty} (-1)^\nu \sin(2\nu+1)\varphi \, J_{2\nu+1}(\lambda\rho) \,. \tag{3.34}$$

Substituting this identity into equation (3.33), we obtain the desired result

$$I_2 = 4\pi i \sum_{\nu=0}^{\infty} (-1)^\nu \sin(2\nu+1)\varphi \int_0^{\infty} F_2(\lambda^2) J_{2\nu+1}(\lambda\rho) d\lambda \,. \tag{3.35}$$

In what follows, it will be convenient to have the range of integration from $-\infty$ to $+\infty$ rather than 0 to ∞. To this end we introduce the following well-known relation between the Bessel and Hankel functions (26, p. 75)

$$J_\nu(z) = \tfrac{1}{2}\left\{ H_\nu^{(1)}(z) - e^{\nu\pi i} H_\nu^{(1)}(-z) \right\} \tag{3.36}$$

which enables us to rewrite (3.31) and (3.35) in the form

$$I_1 = \pi \int_{-\infty}^{\infty} F_1(\lambda^2)\, H_0^{(1)}(\lambda\rho)\,\lambda\, d\lambda \tag{3.37a}$$

and

$$I_2 = 2\pi i \sum_{\nu=0}^{\infty} (-1)^{\nu} \sin(2\nu+1)\varphi \int_{-\infty}^{\infty} F_2(\lambda^2)\, H_{2\nu+1}^{(1)}(\lambda\rho)\, d\lambda\,. \tag{3.37b}$$

We are now ready to rewrite the fundamental integrals U_1 , U_2 , and U_3 in the cylindrical coordinates. We obtain

$$U_1 = \frac{1}{k_0} \int_{-\infty}^{\infty} \frac{e^{i(sh+s_0 z)}}{s+n^2 s_0} H_0^{(1)}(\lambda\rho)\,\lambda\, d\lambda \tag{3.38a}$$

$$U_2 = k_0 \sum_{\nu=0}^{\infty} (-1)^{\nu} \sin(2\nu+1)\varphi \int_{-\infty}^{\infty} \frac{e^{i(sh+s_0 z)}}{s(s+n^2 s_0)} H_{2\nu+1}^{(1)}(\lambda\rho)\, d\lambda \tag{3.38b}$$

and

$$U_3 = k_0 \int_{-\infty}^{\infty} \frac{e^{i(sh+s_0 z)}}{s(s+n^2 s_0)^2} H_0^{(1)}(\lambda\rho)\,\lambda\, d\lambda\,. \tag{3.38c}$$

The integral U_1 can be recognized as one originally found by Sommerfeld in his investigation of the field of a vertical dipole above earth (20). The integrals U_2 and U_3 are somewhat related to the Sommerfeld integrals. The difference comes about from the presence of an "s" multiplying the denominator in both U_2 and U_3 and the factor $(s + n^2 s_0)$ in the denominator of U_2.

We also note for future reference the following transformation of the partial derivatives:

$$\begin{aligned} \partial_x &= \cos\varphi\, \partial_\rho - \frac{\sin\varphi}{\rho}\, \partial_\varphi \\ \partial_y &= \sin\varphi\, \partial_\rho + \frac{\cos\varphi}{\rho}\, \partial_\varphi \end{aligned} \tag{3.39}$$

and the transformation of the components of the fields

$$(\;)_\varphi = -(\;)_x \sin\varphi + (\;)_y \cos\varphi$$
$$(\;)_\rho = (\;)_x \cos\varphi + (\;)_y \sin\varphi \quad . \tag{3.40}$$

3.3d Transformation to spherical coordinates in configuration and transform spaces--For convenience in evaluation of the fundamental integrals by asymptotic methods we shall now transform the previous expressions to spherical coordinates in both the configuration and transform spaces. Thus, we employ the transformation

$$\begin{aligned} z &= r\cos\theta \\ \rho &= r\sin\theta \\ r &= \sqrt{z^2+\rho^2} \end{aligned} \tag{3.41}$$

for the space coordinates and

$$\lambda = k_o \sin\beta \tag{3.42}$$

for the transform variable. As the consequence of the above transformation we note the following:

$$\begin{aligned} s_o &= k_o \cos\beta \\ s &= k_o\sqrt{n^2-\sin^2\beta} \\ s_o z + \lambda\rho &= k_o r\cos(\beta-\theta) \\ d\lambda &= k_o\cos\beta\, d\beta \;. \end{aligned} \tag{3.43}$$

These transformations enable us to recast U_1 , U_2 and U_3 in the following forms:

$$U_1 = \int_{\Gamma_1} \frac{\sin\beta\cos\beta\,\hat{H}_0^{(1)}(k_o\rho\sin\beta)e^{ik_o h\sqrt{n^2-\sin^2\beta}}}{\sqrt{n^2-\sin^2\beta}+n^2\cos\beta}\, e^{ik_o r\cos(\beta-\theta)}\,d\beta \tag{3.44a}$$

$$U_2 = \sum_{\nu=0}^{\infty}(-1)^\nu \sin(2\nu+1)\varphi$$

$$\cdot\int_{\Gamma_1} \frac{\cos\beta\,\hat{H}_{2\nu+1}^{(1)}(k_o\rho\sin\beta)e^{ik_o h\sqrt{n^2-\sin^2\beta}}}{\sqrt{n^2-\sin^2\beta}\left(\sqrt{n^2-\sin^2\beta}+n^2\cos\beta\right)}\, e^{ik_o r\cos(\beta-\theta)}\,d\beta \tag{3.44b}$$

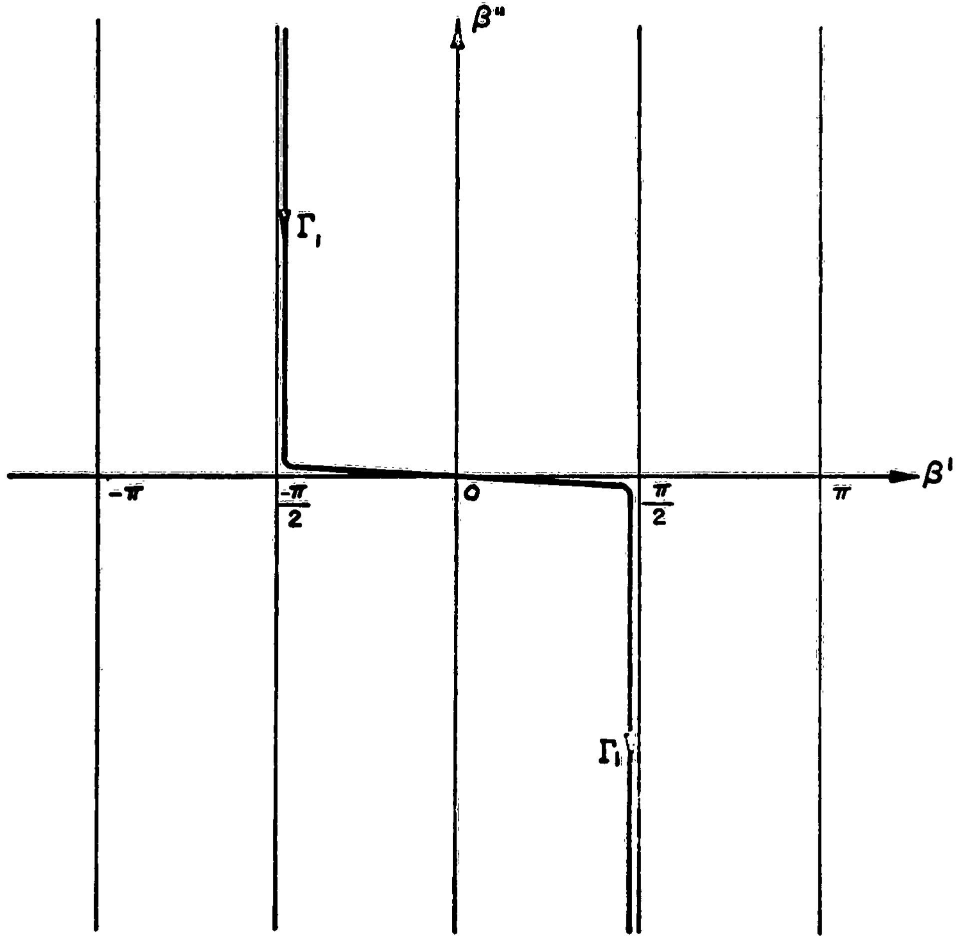

Figure 3.1 - The Path of Integration Γ

and

$$U_3 = \int_{\Gamma_1} \frac{\sin\beta\cos\beta\,\hat{H}_0^{(1)}(k_0\rho\sin\beta)\,e^{ik_0h\sqrt{n^2-\sin^2\beta}}}{\sqrt{n^2-\sin^2\beta}\left(\sqrt{n^2-\sin^2\beta}+n^2\cos\beta\right)^2}\,e^{ik_0r\cos(\beta-\theta)}\,d\beta \tag{3.44c}$$

where we extracted the exponential behavior of the Hankel function and we denoted

$$\hat{H}_m^{(1)}(k_0\rho\sin\beta) = \hat{H}_m^{(1)}(k_0\rho\sin\beta)\;e^{-ik_0\rho\sin\beta}. \tag{3.45}$$

The original path of integration that ran from $-\infty$ to $+\infty$ along the real axis, has now been transformed according to equation (3.42) into the path Γ as shown in Fig. 3.1.

We also note for future reference the following transformation of the partial derivatives:

$$\begin{aligned} \partial_z &= \cos\theta\,\partial_r - \frac{\sin\theta}{r}\,\partial_\theta \\ \partial_\rho &= \sin\theta\,\partial_r + \frac{\cos\theta}{r}\,\partial_\theta \end{aligned} \tag{3.46}$$

and the transformation of the components of the fields

$$\begin{aligned} (\;)_\theta &= -(\;)_z\sin\theta + (\;)_\rho\cos\theta \;, \\ (\;)_r &= (\;)_z\cos\theta + (\;)_\rho\sin\theta \;. \end{aligned} \tag{3.47}$$

Combining the results of (3.46) with those of (3.39) we obtain the transformation of the partial derivatives from the Cartesian system to the spherical system as follows:

$$\begin{aligned} \partial_x &= \cos\varphi\,\sin\theta\,\partial_r + \frac{\cos\varphi\cos\theta}{r}\partial_\theta - \frac{\sin\varphi}{r\sin\theta}\,\partial_\varphi \\ \partial_y &= \sin\varphi\sin\theta\,\partial_r + \frac{\sin\varphi\cos\theta}{r}\partial_\theta + \frac{\cos\varphi}{r\sin\theta}\,\partial_\varphi \\ \partial_z &= \cos\theta\,\partial_r - \frac{\sin\theta}{r}\,\partial_\theta \;. \end{aligned} \tag{3.48}$$

Similarly, combining the results of equations (3.47) and (3.40) we obtain the transformation of the vector components

$$
\begin{aligned}
(\ )_\theta &= \cos\varphi \cos\theta\, (\ )_x + \sin\varphi \cos\theta\, (\ )_y - \sin\theta\, (\ )_z \\
(\ )_\varphi &= -\sin\varphi\, (\ )_x + \cos\varphi\, (\ )_y \\
(\ )_r &= \cos\varphi \sin\theta\, (\ )_x + \sin\varphi \sin\theta\, (\ )_y + \cos\theta\, (\ )_z .
\end{aligned}
\tag{3.49}
$$

3.4 CLOSURE

In the foregoing chapter we applied a high frequency approximation to the rigorous formulation of the present boundary value problem of Chapter 2. This approximation is valid for all wave frequencies appreciably higher than the cyclotron frequency of the plasma. In the configuration space the same approximation is expected to give good results for all polar angles except the one corresponding to the angle of the total internal reflection in the plasma.

Introduction of the high frequency approximation into the expressions for the boundary coefficients resulted in a great simplification of the latter. Subsequently, after the determination of Hertzian vector components in the air, it was found possible to separate the contributions of the plasma anisotropy explicitly. These extra contributions display some heretofore unencountered denominators which will be responsible for some of the unusual phenomena that do not have their counterparts when dealing with isotropic interfaces.

CHAPTER 4

EVALUATION OF THE FUNDAMENTAL INTEGRALS FOR THE DIPOLE IN MAGNETOPLASMA

In the preceding chapter we found the approximate formal solutions for the fields in the air in the form of certain definite integrals. In order to have the results in a form suitable for numerical calculations these integrals have to be evaluated.

The subject integrals do not lend themselves readily to rigorous evaluation and in order to obtain any useful information from them one has to inevitably resort to approximate methods of their evaluation. Fortunately enough, when the field points of interest are at a large distance from the source there is an approximate method available that is known to yield good results. This method is referred to as the "saddle-point method," or the "method of steepest descents" and it was first introduced by Debye for obtaining asymptotic expansion of the Hankel function (11, p. 503). The basic procedure in applying the saddle-point method is as follows. One first locates the stationary or the saddle point of the integrand under consideration. Next, the original path of integration is deformed to a path going through the saddle point. A new path of integration is defined on which the integrand decreases most rapidly when moving away from the saddle point. When this is accomplished the integral has been recast in a form to which the Watson's Lemma is applicable (11, p. 501). One then proceeds expanding the integrand in Taylor's series about the saddle point and integrates term by term. The result is an asymptotic series in terms of the inverse distance from the source.

As we remarked before, the application of the saddle-point method is contingent upon being able to continuously deform the original path of integration to the path of steepest descent passing through the saddle point. This is, of course, always legitimate by Cauchy's theorem providing there are no

singularities between the two paths. If there are singularities between the two paths, a correction to this transformation must be made by adding the contribution from the poles and branch points of the integrand.

The integrands occurring in the electromagnetic problems usually contain a Hankel function of one kind or another. Before the saddle-point method can be applied one must extract the exponential behavior from the Hankel function to determine the location of the saddle point. This is oftentimes done by substituting for the Hankel function its asymptotic formula. Whereas the procedure is basically sound, it introduces an unnecessary singularity and makes the results invalid in the neighborhood of small polar angles. The way to avoid this was devised by Banos and Wesley (2, Ch. 5), who, instead of using the asymptotic formula of the Hankel function, make use of its integral representation which yields a double integral to evaluate rather than a single one. By formal extension of the Watson's Lemma to double integrals, they formed a rigorous basis for the use of the saddle-point method for double integration.

In this thesis we shall avail ourselves of the saddle-point method for double integration of Banos and Wesley (2, Ch. 5) not only for the reason mentioned above but for the following. In our case the basic integral U_2 contains an infinite sum of Hankel functions. Whereas the asymptotic formulas for the Hankel functions of order smaller than the argument are simple, the same is not true in the reverse case. Using the integral representation of the Hankel function in our case avoids then these additional mathematical complications.

4.1 SINGULARITIES IN THE β-PLANE

The evaluation of the basic integrals U_1 , U_2 , and U_3 by the saddle-point method of integration is necessarily contingent upon being able to

deform the original path of integration Γ_1 to the path of steepest descent Γ in Fig. 4.2. In the process of the continuous path deformation we must be sure that any singularities between the two paths are properly accounted for. To this end we shall need to know the exact location of the various singularities of the integrands of U_1, U_2, and U_3 in the complex β-plane.

4.1 a The location of the poles—The examination of the integrands of U_1, U_2 and U_3 in equations (3.44) reveals a possibility of a pole at a point where the denominator vanishes. This occurs where $\beta = \beta_p$ and β_p satisfies the equation

$$\sqrt{n^2 - \sin^2\beta_p} + n^2 \cos\beta_p = 0 \quad . \tag{4.1}$$

The above equation can be readily solved for β_p giving

$$\sin\beta_p = \frac{\pm n}{\sqrt{1+n^2}} \quad . \tag{4.2}$$

To determine the location of this pole in the complex β-plane we assume that n is complex with a small imaginary part. We put

$$n = n' + i\,n'' \; ; \qquad 0 < n'' < n'$$

or (4.3)

$$n = |n|\, e^{i\varphi} \; ; \qquad 0 < \varphi$$

then

$$\sqrt{1+n^2} = M_1 e^{i\theta} \qquad 0 < \theta < \varphi \tag{4.4}$$

For convenience in what follows we shall introduce at this point the concept of Riemann sheets. We shall call the Riemann sheet on which $Im\{(n^2 - \sin^2\beta)^{1/2}\} > 0$ the upper sheet, and the one on which $Im\{(n^2 - \sin^2\beta)^{1/2}\} < 0$ the lower sheet. Consequently, we write for the upper sheet

$$\sqrt{n^2 - \sin^2\beta_p} = \frac{+n^2}{\sqrt{1+n^2}} = \frac{|n|^2 e^{i(2\varphi - \theta)}}{M_1} = M e^{i(2\varphi - \theta)} \tag{4.5a}$$

and

$$\sqrt{n^2 - \sin^2\beta_p} = \frac{-n^2}{\sqrt{1+n^2}} = \frac{-|n|^2 e^{i(2\varphi-\Theta)}}{M_1} = -M e^{i(2\varphi-\Theta)} \qquad (4.5b)$$

for the lower sheet. Thus, in order to have a pole on the upper or lower sheets, the equation (4.1) must be satisfied together with condition (4.5a) or (4.5b) for upper or lower sheet respectively. We obtain

$$M e^{i(\varphi-\Theta)} + |n|^2 e^{i\varphi} \cos\beta_p = 0 \qquad (4.6a)$$

for the upper sheet poles, and

$$-M e^{i(\varphi-\Theta)} + |n|^2 e^{i\varphi} \cos\beta_p = 0 \qquad (4.6b)$$

for the lower sheet poles. The above equation can be put in a more convenient form if we denote $\beta_p = \beta_p' + i\beta_p''$. This gives for the upper sheet

$$\cos\beta_p' \cosh\beta_p'' = -\frac{|n|^2}{M}\cos\Theta \qquad (4.7a)$$

and

$$\sin\beta_p' \sinh\beta_p'' = -\frac{|n|^2}{M}\sin\Theta \qquad (4.7b)$$

From equation (4.7a) it follows that $\pi/2 < \beta' < 3\pi/2$. Using this fact together with equation (4.7b) we arrive at the location of the upper sheet poles as follows:

$$\begin{array}{llll} P_1 : & \pi/2 < \beta_p' < \pi & \beta'' < 0 \\ P_2 : & \pi < \beta_p' < 3\pi/2 & \beta'' > 0 \end{array} \qquad (4.8)$$

For the lower sheet we obtain from (4.6b)

$$\cos\beta_p' \cosh\beta_p'' = \frac{|n|^2}{M}\cos\Theta$$

$$\sin\beta_p' \sinh\beta_p'' = \frac{|n|^2}{M}\sin\Theta$$

which gives the following for the location of the lower sheet poles

$$\begin{array}{llll} P_3 : & 0 < \beta' < \pi/2 & \beta'' > 0 \\ P_4 : & -\pi/2 < \beta' < 0 & \beta' < 0 \end{array} \qquad (4.9)$$

The location of the upper and lower sheet poles is depicted in Fig. 4.1 when $|n|<1$ which is of primary interest to us. We shall have the occasion to return to the subject of pole location later in this thesis when discussing the evaluation of the integrals U_1, U_2, and U_3.

4.1 b The branch point--The integrands of U_1, U_2, and U_3 all contain the radical $(n^2 - \sin^2\beta)^{\frac{1}{2}}$ as a result of which the point $\beta = \beta_s$ where

$$\beta_s = \pm \text{arc} \sin n \tag{4.10}$$

will be a branch point. At each point β the integrands in U_1, U_2, and U_3 can take one two values depending on which sign we choose for the radical. As before, it is convenient here to talk about two sheets of the β-plane (formed by a two-sheeted Riemann surface) on which each integrand is single valued. Then according to our previous convention we call the upper sheet one on which $\text{Im}\left\{(n^2-\sin^2\beta)^{\frac{1}{2}}\right\} > 0$ and the lower sheet one on which $\text{Im}\left\{(n^2-\sin^2\beta)^{\frac{1}{2}}\right\} < 0$. The convergence of the integrals is assured if the path of integration at least begins and ends on the upper sheet. The two sheets will be joined along the lines

$$\text{Im}\left\{(n^2-\sin^2\beta)^{\frac{1}{2}}\right\} = 0 \tag{4.11}$$

starting at the branch points. These lines are shown as dashed lines in Fig. 4.1. It can be shown that the $\text{Im}\left\{(n^2-\sin^2\beta)\right\} > 0$ on the side of the dashed line facing the origin and $\text{Im}\left\{(n^2-\sin^2\beta)\right\} < 0$ on the opposite side.

In the following analysis it will be necessary for us to determine the location of the poles P_3 and P_4 more accurately. In particular it will be essential to know whether the poles P_3 and P_4 lie below or above the cuts issuing from the branch points B_1 and B_2 respectively. For this purpose, in addition to the cut represented by the dashed lines and corresponding to $\text{Im}\left\{(n^2-\sin^2\beta)^{\frac{1}{2}}\right\} = 0$, we construct lines $\text{Re}\left\{(n^2-\sin^2\beta)^{\frac{1}{2}}\right\} = 0$. The latter

lines are shown dotted in Fig. 4.1. The signs of Im $\{(n^2-\sin^2\beta)^{\frac{1}{2}}\}$ and Re $\{(n^2-\sin^2\beta)\}$ change only when one of these lines is crossed. It follows from equation (4.5b) that both signs are negative on the lower sheet. Therefore, the poles P_3 and P_4 are on the same side of the cuts as the origin.

4.2 FORMULATION OF THE CONTRIBUTIONS TO THE FUNDAMENTAL INTEGRALS

In the previous section we determined the singularities of the integrands of U_1, U_2, and U_3. In particular we found a branch point and poles on both Riemann sheets. In this section we shall determine the various contributions to the fundamental integrals as affected by these singularities.

Any one of the basic integrals can be written in the form

$$U = \int_{\Gamma_1} F(\beta)\, \hat{H}_\nu^{(1)}(k_o \rho \sin\beta)\, e^{ik_o r\cos(\beta-\theta)}\, d\beta \tag{4.12}$$

and we wish to evaluate it approximately when r is large. The saddle point of the integrand occurs when the derivative of the exponent vanishes, i.e., when

$$\beta = \theta \tag{4.13}$$

The path of the steepest descents is found from the equation

$$\cos(\beta-\theta) = 1 + i\tau^2 \tag{4.14}$$

where τ is a parameter. It can be readily shown that this path will intersect the real axis at the point $\beta=\theta$, the saddle point, at an angle $\pi/4$ and it will correspond to the path Γ as shown in Fig. 4.2. We now investigate the possibility of replacing the original path of integration Γ_1 with the path Γ corresponding to the path of the steepest descent through the saddle point $\beta=\theta$. Since the poles P_3 and P_4 lie on the lower sheet they do not have to be taken into account when shifting the path of integration from Γ_1 to Γ. The upper sheet poles P_1 and P_2 need not be taken into account either since

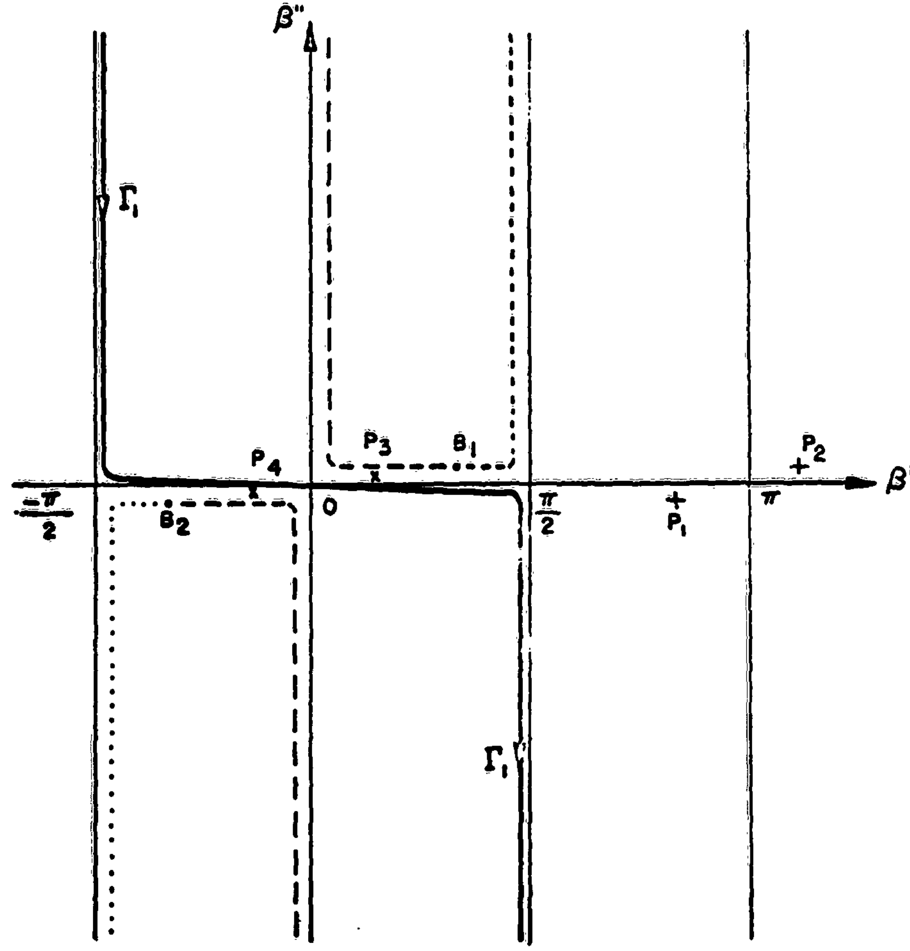

Figure 4.1 - The Complex β -Plane (+ Denotes the Upper Sheet and x Denotes the Lower Sheet Poles, B_1 and B_2 Denote Branch Points When $|n| < 1$)

they will never be crossed lying beyond the region of interest, $-\pi/2 < \theta < \pi/2$. Thus, we are free to replace the original path of integration Γ_1 by the path of the steepest descent Γ.

Now we focus our attention on the following. As we already remarked earlier the integrands of U_1, U_2, and U_3 are double-valued since they contain the radical $(n^2-\sin^2\beta)^{\frac{1}{2}}$. The original path of integration Γ_1 passes over the upper sheet of the corresponding Riemann surface and can be deformed into the path of the steepest descent Γ only when at least the beginning and the end of it lie on this sheet. In the case when the angle of incidence θ does not exceed the angle of the total internal reflection θ_s, i.e., when

$$-\theta_s < \theta < \theta_s \tag{4.15}$$

then in the complex β-plane we have a picture as shown in Fig. 4.2. Here the path of the steepest descent crosses the branch cut twice and the transition from the path Γ_1 to the path Γ is accomplished without any complications. Except for the part shown dotted in Fig. 4.2, the path Γ will lie in the upper sheet. In this case the only contribution to the integrals U_1, U_2, and U_3 will be the one from the saddle point.

The situation is different when the angle of incidence, θ, exceeds that of the total internal reflection, θ_s. Here the path of the steepest descent will cross the branch cut only once, crossing from the upper sheet to the lower sheet and going to infinity along the lower sheet, without connecting anywhere with the path Γ_1 which is inadmissible. We can, however, construct a more complicated path of integration, supplementing the path of the steepest descent Γ by a contour encompassing the cut in such a way that the beginning and the end of the more complicated path will lie on the upper sheet.

To this end we introduce "cuts" in the complex β-plane along the branch cut, i.e., along the line on which $\mathrm{Im}\left\{(n^2-\sin^2\beta)^{\frac{1}{2}}\right\} = 0$ which are marked

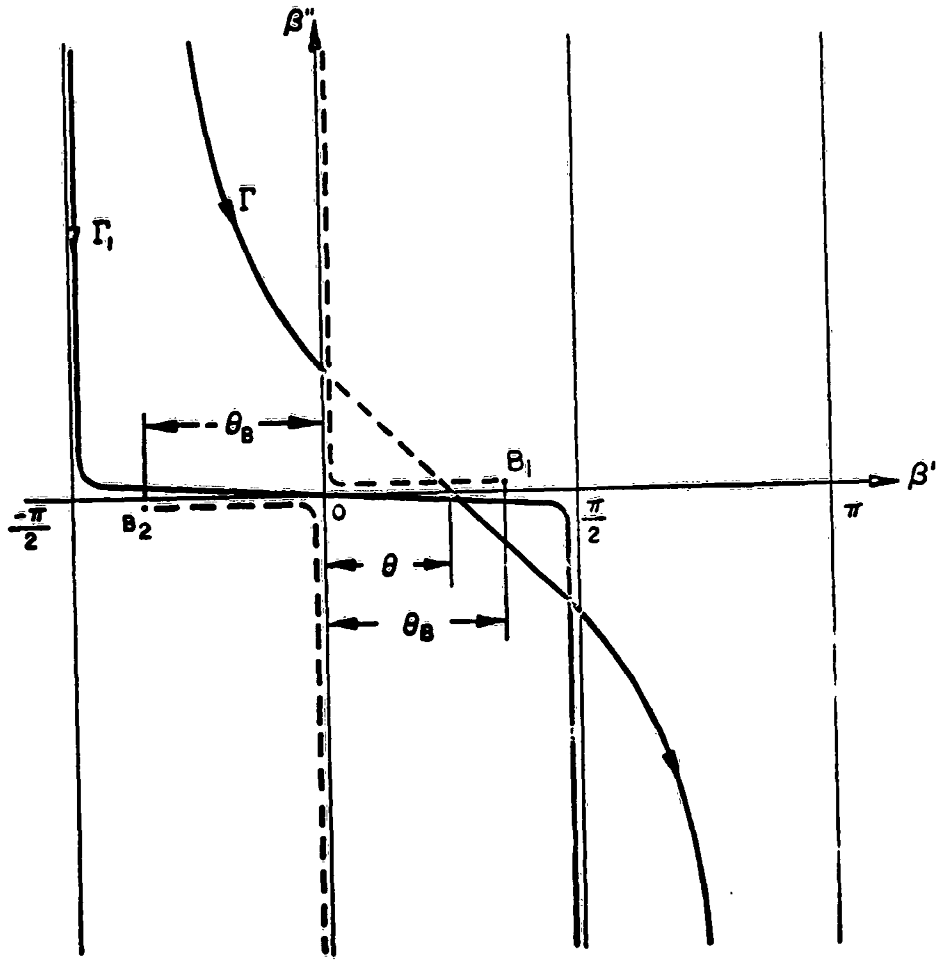

Figure 4.2 - Deformed Path of Integration For $-\theta_B < \theta < \theta_B$

dotted in Fig. 4.1 and 4.2. The equation of the cuts will be

$$n^2 - \sin^2\beta = x^2 \qquad 0 \leq x^2 \leq \infty \tag{4.16}$$

and according to the previous analysis $\mathrm{Im}\{(n^2-\sin^2\beta)^{\frac{1}{2}}\} > 0$ on the sides of the cuts facing the origin and $\mathrm{Im}\{(n^2-\sin^2\beta)^{\frac{1}{2}}\} < 0$ on the other side.

For $\theta > 0$ this new path of integration which was first proposed by Ott (18, p. 451) runs along the line path of the steepest descents Γ then along the borders of the cut, intersects the branch cut once thus entering the lower sheet. It then continues along the path Γ and intersecting the branch cut once more it ends on the upper sheet. The remainder of the path is along the path Γ on the upper sheet. The situation is depicted in Fig. 4.3. For $\theta < 0$ the situation is analogous.

As a result the complete expression for any of the integrals U_1, U_2, or U_3 will consist of two parts and can be written

$$U = U^{(s)} + U^{(B)} u\,(\theta - \theta_B) \tag{4.17}$$

where $U^{(s)}$ will stand for the contribution from the saddle point and $U^{(B)}$ will stand for the contribution from the integration about the borders of the branch cut. Since the contribution from the branch cut does not appear until $|\theta| > |\theta_s|$, we have employed a unit step function $u(\theta - \theta_B)$ to denote this.

<u>4.2 a Formulation of the contribution from the saddle point</u>—As we remarked earlier each one of the fundamental integrals can be written in the form

$$U = \int_{\Gamma} F(\beta)\, \hat{H}_{\nu}^{(1)}(k_o \rho \sin\beta)\, e^{ik_o r\cos(\beta-\theta)}\, d\beta \tag{4.18}$$

and the saddle point of the integrand is located where the derivative of the exponent vanishes, i.e., where $\beta = \theta$. First we put

$$w = \beta - \theta \tag{4.19}$$

which transfers the saddle point to the point $w = 0$ in the complex w-plane.

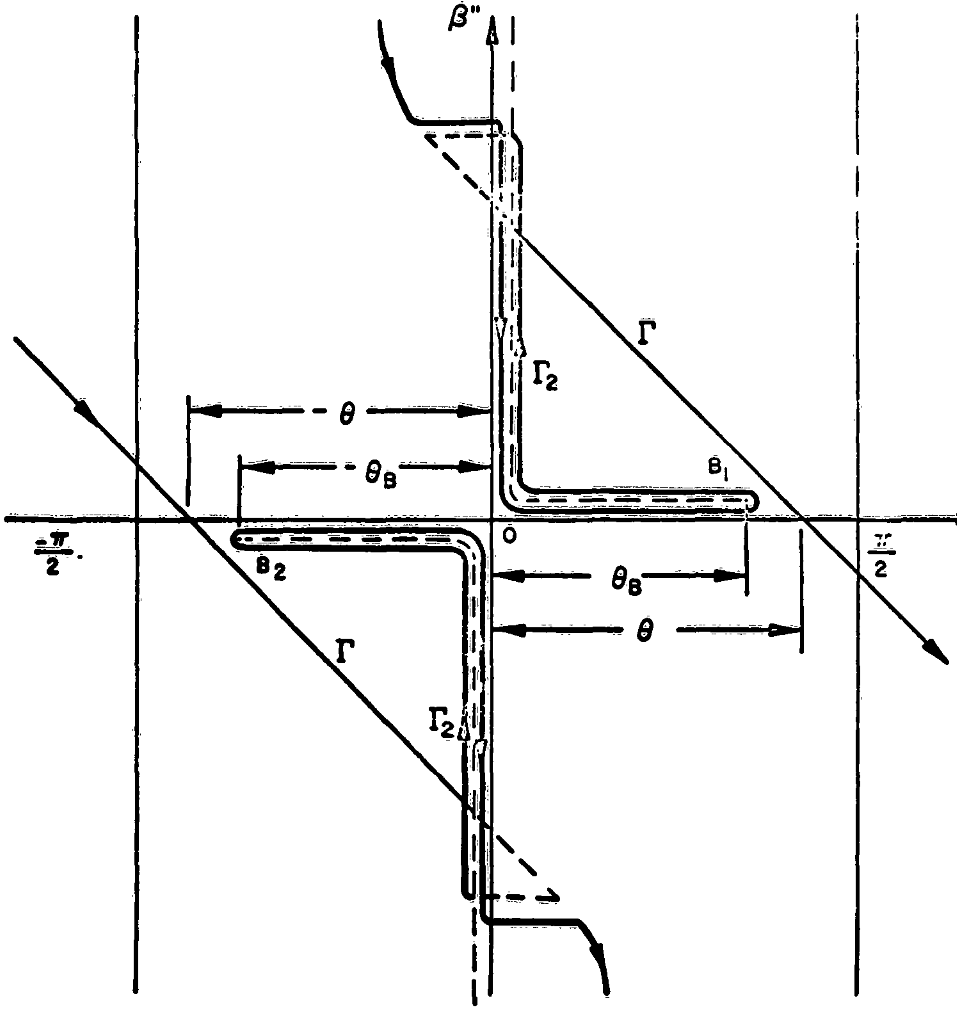

Figure 4.3 - Deformed Path of Integration For $|\theta| > |\theta_B|$

Following the procedure of Banos and Wesley (2, p. 97) we make the next transformation as follows:

$$\frac{x^2}{2} = ik_o r\,(1 - \cos w) \tag{4.20}$$

noting that

$$\frac{dw}{dx} = \frac{x}{i\,k_o r \sin w}\,. \tag{4.21}$$

We also note that the right-hand side of equation (4.20) can be expanded into a power series as follows:

$$\begin{aligned} \frac{x^2}{2} &= i\,k_o r\,w^2 \left\{ \frac{1}{2} - \frac{w^2}{4!} + \frac{w^4}{6!} + \cdots \right\} \\ &= w^2 \left\{ c_o + c_2 + c_4 + \cdots \right\}. \end{aligned} \tag{4.22}$$

The above series can be inverted (2, p. 134), giving

$$w = \frac{x}{(ik_o r)^{\frac{1}{2}}} + \frac{1}{24}\,\frac{x^3}{(ik_o r)^{3/2}} + \cdots \tag{4.23}$$

From the above we note that w is an odd function of x, $w(x) = -w(-x)$. Thus, as x changes sign, so does w.

Now we can transform the integral in equation (4.18) to

$$U^{(s)} = e^{ik_o r} \int_0^{\infty} G(x)\, e^{-x^2/2}\, dx \tag{4.24}$$

where

$$G(x) = \left\{ F(w+\theta)\hat{H}_\nu^{(1)}(k_o\rho \sin(w+\theta)) + F(-w+\theta)\hat{H}_\nu^{(1)}(k_o\rho\sin(-w+\theta)) \right\} \frac{dw}{dx} \tag{4.25}$$

Putting this expression under the integral sign in (4.24) results in two integrals with the integration range from 0 to $+\infty$. In the second integral we make the substitution $x = -x'$ and reverse the limits. Due to the fact that dw/dx is an even function of x, we can add these two integrals to form a single one as follows:

$$U^{(s)} = \frac{e^{ik_o r}}{ik_o r} \int_{-\infty}^{\infty} P(x)\, \hat{H}_\nu^{(1)}(k_o\rho \sin(w+\theta))\, e^{-x^2/2}\, dx \tag{4.26}$$

where

$$P(x) = \frac{x F(w+\theta)}{\sin w} \tag{4.27}$$

Now we make use of the integral representation of the Hankel function (26, p. 196)

$$H_\nu^{(1)}(z) = \frac{4e^{-i\nu\pi}e^{iz}}{\sqrt{\pi}\, 2^{3\nu}\Gamma(\nu+\frac{1}{2}) z^\nu}\int_0^\infty y^{2\nu}(4iz-y^2)^{\nu-\frac{1}{2}}e^{-y^2/2}\,dy \tag{4.28}$$

which now we put into equation (4.26) to obtain

$$U^{(s)} = \frac{4e^{i(k_0 r-\nu\pi)}}{ik_0 r\sqrt{\pi}\, 2^{3\nu}\Gamma(\nu+\frac{1}{2})(k_0\rho)^\nu}\int_{-\infty}^{\infty}\int_0^\infty Q(x,y)\, e^{-x^2/2}\, e^{-y^2/2}\,dx\,dy \tag{4.29}$$

where

$$Q(x,y) = \frac{x y^{2\nu} F(w+\theta)\left[4ik_0\rho\sin(w+\theta)-y^2\right]^{\nu-\frac{1}{2}}}{\sin w\left[\sin(w+\theta)\right]^\nu} \tag{4.30}$$

We recognize the integral (4.29) as being analogous in form to single integral of the type to which Watson's Lemma can be applied. A theoretical basis extending the Watson's Lemma to double integrals has been provided by Banos and Wesley (2, Ch.5). Thus, at this point all that remains to be done to find the saddle-point contribution to the integral U, is to expand the integrand Q(x,y) in a double power series and integrate term by term. When performing such an integration we shall avail ourselves of the following well-known integrals (9, p. 64)

$$\int_{-\infty}^{\infty} e^{-ax^2/2}\,dx = \sqrt{\frac{2\pi}{a}} \tag{4.31a}$$

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2/2}\,dx = \sqrt{\frac{2\pi}{a^3}} \tag{4.31b}$$

and in general

$$\int_{-\infty}^{\infty} x^{2\nu} e^{-ax^2/2}\, dx = \Gamma(\nu+\tfrac{1}{2})\left(\frac{2}{a}\right)^{\nu+\frac{1}{2}} = \frac{(2\nu)!\sqrt{2\pi}}{\nu!\,2^{\nu} a^{\nu+\frac{1}{2}}} \tag{4.31c}$$

4.2 b Formulation of the contribution from the branch cut (5,p.270)—To find the contribution to the integral (4.12) due to the integration about the borders of the cuts, we first consider the case of $\theta > \theta_B > 0$ corresponding to the upper cut. We write

$$U^{(B)} = \int_{\Gamma_2} F(\beta)\, \hat{H}_{\nu}^{(1)}(k_0 \rho \sin\beta)\, e^{ik_0 r\cos(\beta-\theta)}\, d\beta \tag{4.32}$$

and integrate formally on both sides of the cut, getting

$$U^{(B)} = \int_{i\infty}^{B_1} F_+(\beta)\, \hat{H}_{\nu}^{(1)}(k_0 \rho \sin\beta)\, e^{ik_0 r\cos(\beta-\theta)}\, d\beta$$

$$+ \int_{B_1}^{i\infty} F_-(\beta)\, \hat{H}_{\nu}^{(1)}(k_0 \rho \sin\beta)\, e^{ik_0 r\cos(\beta-\theta)}\, d\beta \tag{4.33}$$

where $F_+(\beta)$ is the value of the function $F(\beta)$ on the left side of the cut and $F_-(\beta)$ is the value of the same function on the right side of the cut. Interchanging the limits of integration in the first integral we obtain

$$U^{(B)} = \int_{B_1}^{i\infty} R(\beta)\, \hat{H}_{\nu}^{(1)}(k_0 \rho \sin\beta)\, e^{ik_0 r\cos(\beta-\theta)}\, d\beta \tag{4.34}$$

where

$$R(\beta) = F_-(\beta) - F_+(\beta)\,. \tag{4.35}$$

We note that since the imaginary part of $(n^2 - \sin^2\beta)^{\frac{1}{2}}$ is zero on the cut and the real part has different signs on the two borders of the cut, then $F_+(\beta)$ and $F_-(\beta)$ will differ from one another only in the sign of this square root.

In what follows it will be convenient to deform the path of integration in such a way that it goes from the branch point B_1 along the line on which

the exponent in the integrand decreases most rapidly. This will be the line on which

$$\mathrm{Re}\{\cos(\beta-\theta)\} = \mathrm{const} \tag{4.36}$$

Then it is necessary that

$$\mathrm{Im}\{\cos(\beta-\theta)\} > 0 \tag{4.37}$$

We assume for convenience that n is real. Then setting $\beta = \theta_s$ in equation (4.36) gives

$$\mathrm{const} = \cos(\theta_s - \theta) \tag{4.38}$$

Now since

$$\cos(\beta-\theta) = \cos(\beta'-\theta)\cosh\beta'' + i\sin(\theta-\beta')\sinh\beta'' \tag{4.39}$$

we obtain for the path of integration from equations (4.36) and (4.38)

$$\cos(\beta'-\theta)\cosh\beta'' = \cos(\theta_s-\theta) \; . \tag{4.40}$$

This path is shown in Fig. 4.4 as a solid line. The condition (4.37) is also satisfied on this path. The deformation from the path Γ_2 to Γ_2' presents no difficulty since there are no singularities between them. In particular, the pole P_3 lies below and not above the line Γ_2 as we remarked earlier in Section 4.1.

In view of the results of equation (4.40) we rewrite equation (4.34)

$$U^{(R)} = e^{ik_o r\cos(\theta-\theta_s)} \int_{\Gamma_2'} R(\beta)\,\hat{H}_\nu^{(1)}(k_o\rho\sin\beta)\, e^{-k_o r\sin(\theta-\beta')\sinh\beta''} d\beta \, . \tag{4.41}$$

Now using equation (4.40) we express the entire integrand in terms of β'' which we treat as a new variable of integration. The new limits of integration are from 0 to ∞ . For $k_o r$ large on the path of the steepest descent only small values of β'' will be important. Moreover, we can set $\beta' = \theta$ on the initial part of the path. Thus,

$$\sin(\theta-\beta')\sinh\beta'' \sim \sin(\theta-\theta_s)\,\beta'' \tag{4.42a}$$

and

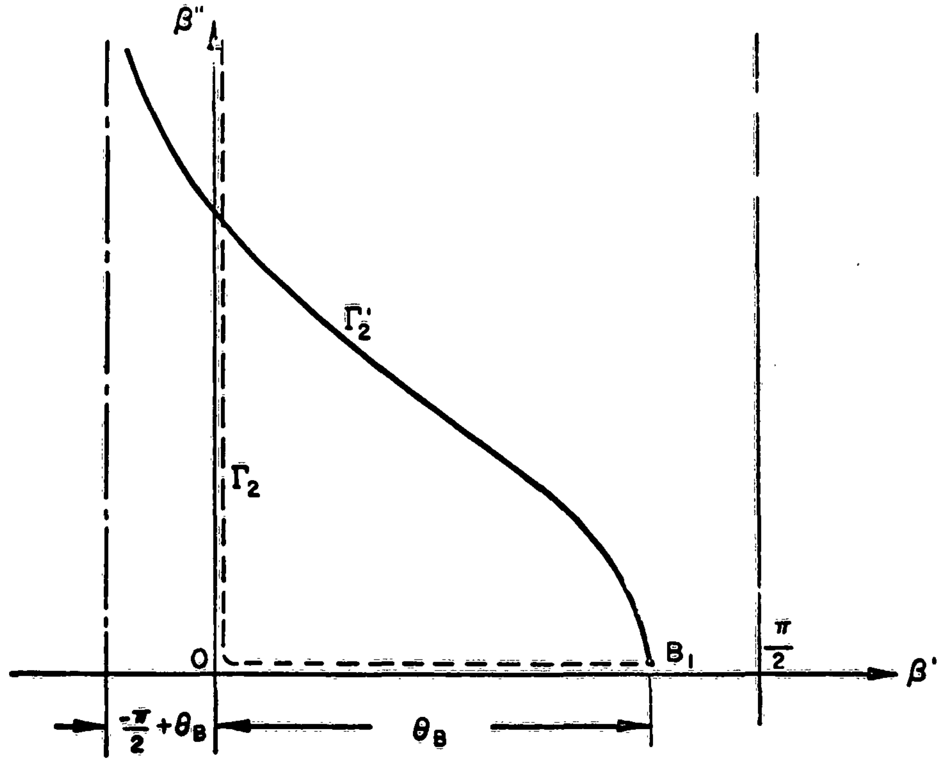

Figure 4.4 - Deformed Path For Branch-Cut Integration When $\theta_B < \theta < \pi/2$

$$d\beta = i\, d\beta'' \tag{4.42b}$$

Finally, we put

$$\beta'' = \frac{x^2}{2} \tag{4.43}$$

and rewrite (4.41)

$$U^{(\theta_s)} = i\, e^{ik_o r\cos(\theta-\theta_s)} \int_0^\infty x\, R(\beta)\, \hat{H}_\nu^{(1)}(k_o \rho \sin\beta)\, e^{-k_o r \sin(\theta-\theta_s) x^2/2}\, dx \tag{4.44}$$

where β in the above equation stands for

$$\beta = \theta_s + \frac{i x^2}{2} \tag{4.45}$$

Next we perform the integration around the borders of the branch cut originating at B_2 pertinent to angles θ such that $-\pi/2 < \theta < \theta_s$. Integrating formally we get

$$U^{(\theta_s)} = \int_{-i\infty}^{B_2} F_-(\beta)\, \hat{H}_\nu^{(1)}(k_o \rho \sin\beta)\, e^{ik_o r\cos(\beta-\theta)} d\beta$$

$$+ \int_{B_2}^{-i\infty} F_+(\beta)\, \hat{H}_\nu^{(1)}(k_o \rho \sin\beta)\, e^{ik_o r\cos(\beta-\theta)} d\beta \tag{4.46}$$

where again $F_+(\beta)$ is the value of the function $F(\beta)$ on the left side of the cut and $F_-(\beta)$ is the value of the same function on the right side of the cut. Interchanging the limits of integration in the first integral, we obtain

$$U^{(\theta_s)} = -\int_{B_2}^{-i\infty} R(\beta)\, \hat{H}_\nu^{(1)}(k_o \rho \sin\beta)\, e^{ik_o r\cos(\beta-\theta)} d\beta \tag{4.47}$$

where $R(\beta)$ is the same as in equation (4.35). Proceeding as before, we define a new path of integration on which $\mathrm{Re}\{\cos(\beta-\theta)\} = \text{const}$. But this time we have to set $\beta = -\theta_s$ in equation (4.36) to obtain

$$\text{const} = \cos(\theta_s + \theta) \tag{4.48}$$

and the new path of integration is expressed by

$$\cos(\beta' - \theta)\cosh\beta'' = \cos(\theta_s + \theta)\ . \tag{4.49}$$

This path is shown in Fig. 4.5 as a solid line. In view of the above results we rewrite equation (4.47) as follows:

$$U^{(B_2)} = - e^{ik_0 r\cos(\theta_s+\theta)} \int_{\Gamma_2'} R(\beta)\, \hat{H}_\nu^{(1)}(k_0\rho\sin\beta)\, e^{-k_0 r\sin(\theta\ \beta')\sinh\beta''} d\beta. \quad (4.50)$$

Now using equation (4.49) we express the entire integrand in terms of β'' which we treat as a new variable of integration. The new limits of integration are from 0 to $-\infty$. Moreover, we set $\beta' = -\theta_s$ on the initial path. Thus,

$$\sin(\theta - \beta')\ \sinh\beta'' \sim \sin(\theta_s + \theta)\,\beta'' \quad (4.51a)$$

and

$$d\beta = i\, d\beta'' . \quad (4.51b)$$

Now we rewrite (4.50)

$$U^{(B_2)} = - i e^{ik_0 r\cos(\theta_s+\theta)} \int_0^{-\infty} R(\beta)\hat{H}_\nu^{(1)}(k_0\rho\sin\beta)\, e^{-k_0 r\sin(\theta_s+\theta)\beta''} d\beta''. \quad (4.52)$$

It will be more convenient to have the range of integration from 0 to $+\infty$ rather than, from 0 to $-\infty$; to this end we put

$$\beta'' = -\frac{x^2}{2} \quad (4.53)$$

which gives

$$U^{(B_2)} = i e^{ik_0 r\cos(|\theta|-\theta_s)} \int_0^{\infty} xR(-\beta)\hat{H}_\nu^{(1)}(-k_0\rho\sin\beta) e^{-k_0 r\sin(|\theta|-\theta_s)x^2/2} dx \quad (4.54)$$

where β stands for the same as in equation (4.45). Noting that

$$\rho_{\theta \gtrless 0} = \pm r\sin|\theta| \quad (4.55)$$

we can include (4.41) and (4.55) in a single equation as follows:

$$U^{(B)}_{\theta\gtrless 0} = i e^{ik_0 r\cos(|\theta|-\theta_s)} \int_0^{\infty} xR(\pm\beta)\hat{H}_\nu^{(1)}(k_0 r\sin|\theta|\ \sin\beta)$$

$$\cdot\ e^{-k_0 r\sin(|\theta|-\theta_s)x^2/2} dx . \quad (4.56)$$

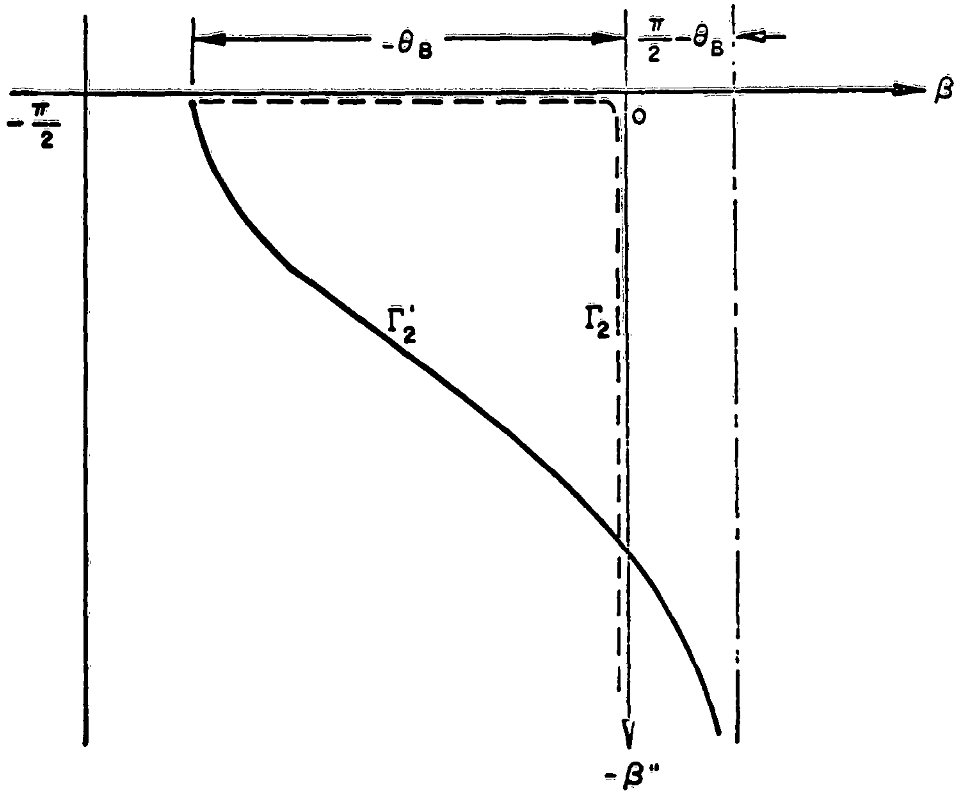

Figure 4.5 - Deformed Path For Branch-Cut Integration When $-\pi/2 < \theta < \theta_B$

Now we insert the integral representation of the Hankel function and obtain finally

$$U^{(B)}_{\theta \gtrless 0} = \frac{i4e^{-i\nu\pi}e^{ik_o r\cos(|\theta|-\theta_B)}}{\sqrt{\pi}\,2^{3\nu}\Gamma(\nu+1/2)}$$

$$\cdot\int\!\!\int_0^{\infty} \frac{xy^{2\nu}R(\pm\beta)[4ik_o r\sin|\theta|\sin\beta - y^2]^{\nu-1/2}}{(k_o r\sin|\theta|\sin\beta)^{\nu}} e^{-y^2/2 - k_o r\sin(|\theta|-\theta_B)x^2/2}\,dx\,dy \quad (4.57)$$

We pause for a moment to think about the significance of the above results.

In the formulation of the branch-cut integration we introduced a change of the variable in the form

$$\beta = \theta_B + \frac{ix^2}{2} \quad (4.58)$$

and subsequently concluded that only small values of x would contribute to the integral. While this statement in itself is true, it produces an undesirable complication. For if β is given by (4.58) then we are forced to integrate in a small neighborhood about s = 0 which is just the point where the high frequency approximation (3.7) is not valid.

Whether the results produced by such an integration would still make sense, it is difficult to appraise at this time. However, in a more rigorous analysis which we are able to carry out in the case of the line source problems, Section 9.6, indicate that the results of such an integration are indeed not valid since they produce expressions substantially different from those that are obtained when the high frequency approximation is introduced after integration.

We shall, henceforth, neglect the contribution from the branch-cut integration altogether in the present problem. First, because we are not able to

effectively carry it out in the present formulation and second, as we shall see later in Section 9.6, such a branch-cut integration shows that the leading terms of the lateral field are of second order only.

4.3 EVALUATION OF THE FUNDAMENTAL INTEGRALS AT THE SADDLE POINT

In Section 4.2 we derived a general formula for evaluation of the saddle-point contribution to the fundamental integrals. In this section we shall apply these formulas to find the asymptotic expressions for U_1, U_2, and U_3.

4.3 a The integral U_1 —Applying equations (4.29) and (4.30) to the expression for U_1 in equation (3.44a) we obtain

$$U_1 = \frac{4e^{ik_0 r}}{ik_0 r\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} Q_1(x,y)\, e^{-x^2/2}\, e^{-y^2/2}\, dx\, dy \tag{4.59}$$

where

$$Q_1(x,y) = \frac{x\sin(w+\theta)\cos(w+\theta)e^{ik_0 h\sqrt{n^2-\sin^2(w+\theta)}}}{\sin w\sqrt{4ik_0\rho\sin(w+\theta)-y^2}\left[\sqrt{n^2-\sin^2(w+\theta)}+n^2\cos(w+\theta)\right]} \tag{4.60}$$

and $w(x)$ is given by equation (4.23). We find for the leading term

$$Q_1(x,y) \sim \frac{1}{2}\cdot\frac{\cos\theta\, e^{ik_0 h\sqrt{n^2-\sin^2\theta}}}{n^2\cos\theta+\sqrt{n^2-\sin^2\theta}}\,. \tag{4.61}$$

Putting this into (4.59) and using the integrals in (4.31) we obtain

$$U_1 \sim \frac{-i2e^{ik_0 r}}{k_0 r}\cdot\frac{\cos\theta\, e^{ik_0 h\sqrt{n^2-\sin^2\theta}}}{n^2\cos\theta+\sqrt{n^2-\sin^2\theta}}\,. \tag{4.62}$$

4.3 b The integral U_2 —Applying equations (4.29) and (4.30) to the expression for U_2 in equation (3.44b) we obtain

$$U_2 = -\sum_{\nu=0}^{\infty}\frac{(-1)^\nu\sin(2\nu+1)\varphi\, e^{ik_0 r}}{2^{6\nu+1}\sqrt{\pi}\, ik_0 r(k_0\rho)^{2\nu+1}\Gamma(2\nu+3/2)}\int_{-\infty}^{\infty}\int_{0}^{\infty}Q_2(x,y)\,e^{-x^2/2}\,e^{-y^2/2}\,dx\,dy \tag{4.63}$$

where

$$Q_2(x,y)= \frac{x y^{2(2\nu+1)} \cos(w+\theta) e^{ik_o h\sqrt{n^2-\sin^2(w+\theta)}}\left[4ik_o \rho \sin(w+\theta)-y^2\right]^{2\nu+\frac{1}{2}}}{\sin w\left[\sin(w+\theta)\right]^{2\nu+1}\sqrt{n^2-\sin^2(w+\theta)}\left[\sqrt{n^2-\sin^2(w+\theta)}+n^2\cos(w+\theta)\right]} \tag{4.64}$$

and $w(x)$ is given by equation (4.23). We find for the leading term

$$Q_2(x,y)\sim \frac{2^{4\nu+1}(ik_o r)^{2\nu+1}\cos\theta \sin^{2\nu}\theta\, e^{ik_o h\sqrt{n^2-\sin^2\theta}}}{\sqrt{n^2-\sin^2\theta}\,(\sqrt{n^2-\sin^2\theta}+n^2\cos\theta)}\, y^{2(2\nu+1)} \tag{4.65}$$

Putting this into equation (4.63) gives

$$U_2 = -\sum_{\nu=0}^{\infty} \frac{\sin(2\nu+1)\varphi\, e^{ik_o r\cos\theta}\, e^{ik_o h\sqrt{n^2-\sin^2\theta}}}{2^{2\nu}\sqrt{\pi}\, k_o r\sin\theta\,\Gamma(2\nu+3/2)\sqrt{n^2-\sin^2\theta}\,(\sqrt{n^2-\sin^2\theta}+n^2\cos\theta)}$$

$$\cdot\int_{-\infty}^{\infty}\int_{0}^{\infty} y^{2(2\nu+1)}\, e^{-x^2/2-y^2/2}\, dx\, dy\,. \tag{4.66}$$

Using (4.31) we note

$$\int_{-\infty}^{\infty} y^{2(2\nu+1)} e^{-y^2/2}\, dy = \Gamma(2\nu+3/2)\, 2^{2\nu+3/2}$$

$$\int_{0}^{\infty} e^{-x^2/2}\, dx = \sqrt{\frac{\pi}{2}} \tag{4.67}$$

which upon substitution into (4.66) gives

$$U_2 = -\frac{2\cos\theta\, e^{ik_o h\sqrt{n^2-\sin^2\theta}}\, e^{ik_o r}}{k_o r\sin\theta\sqrt{n^2-\sin^2\theta}\,(\sqrt{n^2-\sin^2\theta}+n^2\cos\theta)}\cdot\sum_{\nu=0}^{\infty}\sin(2\nu+1)\varphi\,. \tag{4.68}$$

The series on the extreme right in the above equation can be summed. We write formally

$$\sum_{\nu=0}^{\infty}\sin(2\nu+1)\varphi = \mathrm{Im}\left\{e^{i\varphi}\sum_{\nu=0}^{\infty} e^{i2\nu\varphi}\right\}$$

and recognize the series on the right as the binomial series consisting of terms $(e^{i2\varphi})^{\nu}$. Thus, we can sum immediately by writing

$$\sum_{\nu=0}^{\infty}\sin(2\nu+1)\varphi = \mathrm{Im}\left\{\frac{e^{i\varphi}}{1-e^{i2\varphi}}\right\} = \frac{1}{2\sin\varphi}\,. \tag{4.69}$$

Inserting the above result in equation (4.68) we obtain finally

$$U_2 \sim \frac{-\cos\theta\, e^{ik_0 r} e^{ik_0 h\sqrt{n^2-\sin^2\theta}}}{k_0 r \sin\theta \sin\varphi \sqrt{n^2-\sin^2\theta}\,(\sqrt{n^2-\sin^2\theta}+n^2\cos\theta)} \qquad (4.70)$$

4.3 c The integral U_3.—Applying equations (4.29) and (4.30) to the expression for U_3 in equation (3.44c) we obtain

$$U_3 = \frac{4e^{ik_0 r}}{ik_0 r \pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} Q_3(x,y)\, e^{-x^2/2}\, e^{-y^2/2}\, dxdy \qquad (4.71)$$

where

$$Q_3(x,y) = \frac{x\sin(w+\theta)\cos(w+\theta)\left[4ik_0\sin(w+\theta)-y^2\right]^{-1/2} e^{ik_0 h\sqrt{n^2-\sin^2(w+\theta)}}}{\sin w\sqrt{n^2-\sin^2(w+\theta)}\left[\sqrt{n^2-\sin^2(w+\theta)}+n^2\cos(w+\theta)\right]^2} \qquad (4.72)$$

where $w(x)$ is given by equation (4.23). We obtain for the leading term

$$Q_3(x,y) \sim \frac{\cos\theta\, e^{ik_0 h\sqrt{n^2-\sin^2\theta}}}{2\sqrt{n^2-\sin^2\theta}\,(\sqrt{n^2-\sin^2\theta}+n^2\cos\theta)^2}\ . \qquad (4.73)$$

Substituting the last result into (4.71) and making use of the integrals in (4.31), we obtain

$$U_3 \sim \frac{-i2\cos\theta\, e^{ik_0 r} e^{ik_0 h\sqrt{n^2-\sin^2\theta}}}{k_0 r\sqrt{n^2-\sin^2\theta}\,(\sqrt{n^2-\sin^2\theta}+n^2\cos\theta)^2}\ . \qquad (4.74)$$

4.4 DIFFERENTIABILITY OF THE ASYMPTOTIC FORMS

The usefulness of the asymptotic forms for U_1, U_2, and U_3 found in the previous section is necessarily contingent upon whether these forms are differentiable or not. In this section we shall show that our asymptotic forms are differentiable by the way of showing that the differential relations existing among their integral representations are also satisfied by their asymptotic forms.

It can be shown that the following differential relationships are

satisfied by the integral representations of U_1, U_2, and U_3:

$$U_1 = -\frac{2i}{k_o^2} \partial_y \partial_h U_2 \tag{4.75a}$$

$$2i \partial_y U_2 = (\partial_h + n^2 \partial_z) U_3 . \tag{4.75b}$$

In what follows we shall show that the above relationships are satisfied by the asymptotic forms of the contributions from the saddle point.

To demonstrate the differentiability of U_1, U_2, and U_3, we first note using (3.48) and the results of Section 4.3;

$$\partial_x(\;) \sim i k_o \cos\varphi \sin\Theta (\;) \tag{4.76a}$$

$$\partial_y(\;) \sim i k_o \sin\varphi \sin\Theta (\;) \tag{4.76b}$$

$$\partial_z(\;) \sim i k_o \cos\Theta (\;) \tag{4.76c}$$

$$\partial_h(\;) \sim i k_o \sqrt{n^2 - \sin^2\Theta} (\;) \tag{4.76d}$$

where the first three relationships are valid only to the first inverse order of r. Applying these to (4.75a) and (4.75b), we obtain

$$U_1 = 2i \sin\varphi \sin\Theta \sqrt{n^2 - \sin^2\Theta}\; U_2 \tag{4.77a}$$

$$2i \sin\varphi \sin\Theta\, U_2 = (\sqrt{n^2 - \sin^2\Theta} + n^2 \cos\Theta) U_3 . \tag{4.77b}$$

An examination of (4.62) and (4.68) reveals that the relationship (4.77a) is satisfied. Similarly, an examination of (4.68) and (4.74) reveals that the relationship (4.77b) is satisfied.

This concludes the proof that U_1, U_2, and U_3 are differentiable to the first inverse order in r.

4.6 CLOSURE

In the foregoing chapter we evaluated approximately the fundamental integrals U_1, U_2, and U_3. In the process of their evaluation we found that the asymptotic representation of each of these integrals consists of two basic parts. The first part comes from the contribution of the saddle point of the integrand and represents the well-known radiation field with the leading term characterized by the factor $r^{-1} \exp(ik_o r)$. The second part which appears for $\theta > \theta_s$ where θ_s corresponds to the angle of the total internal reflection in the plasma, comes from the integration along the borders of the branch cut and represents the lateral field (5, p. 270). While the contribution of the branch-cut integration was formulated, it was not used to find the actual components of the lateral field. The reason for this was the fact that the main contribution to the lateral field appeared to have come from the point where the earlier introduced high frequency approximation was not valid. The subject of lateral waves will be discussed later, however, when dealing with the line sources where rigorous evaluation of the lateral field is possible.

Finally, we remark that the asymptotic representations of the fundamental integrals U_1, U_2, and U_3 are differentiable and thus are valid representations and as such they can be used to find the asymptotic form of the components of the Hertzian vector and subsequently the field components in the radiation zone.

CHAPTER 5

FIELDS AND POWER FLOW IN THE AIR FOR THE DIPOLE IN MAGNETOPLASMA

In the preceding chapter we found the leading terms of the asymptotic representation for the fundamental integrals U_1 , U_2 , and U_3 . In addition we have demonstrated that the asymptotic forms were differentiable to the desired order and as such they could be used to find first, asymptotic forms of the components of the Hertzian vector and second, the asymptotic forms of the field components themselves.

In what follows we shall use these facts to arrive at the results of this chapter.

5.1 THE HERTZIAN VECTOR AND THE FIELDS

In this section we first find the asymptotic forms for the components of the Hertzian vector in spherical coordinates. Following that, we express the spherical components of the fields in terms of the components of the Hertzian vector and finally find the explicit forms of the various field components in spherical coordinates.

<u>5.1 a The Hertzian vector</u>—Applying the expressions for the partial derivatives from the results of (4.76), we rewrite (3.27a) and (3.27b) obtaining

$$\Pi_{x0} = -\frac{n^2 m k_0}{4\pi\omega\mu_0}\Big\{U_1 + i\kappa\sin\theta\Big[\sin\varphi\sqrt{n^2-\sin^2\theta}\,U_3 + i\cos^2\varphi\sin\theta\Big(\frac{\cos\theta}{\cos\theta+\sqrt{n^2-\sin^2\theta}} + ik_0h\sqrt{n_1^2-\sin^2\theta}\Big)U_2\Big\} \qquad (5.1a)$$

and

$$\Pi_{zo} = -\frac{n^2 m k_o \cos\varphi \sin\theta}{4\pi\omega\mu_o}\left\{\frac{\cos\theta - \sqrt{n^2-\sin^2\theta}}{n^2}U_1 + \kappa\left[-i\sin\varphi\sin\theta\, U_3\right.\right. \tag{5.1b}$$

$$\left.\left. - \frac{\sin^2\varphi\sin^2\theta + \cos^2\theta}{\cos\theta + \sqrt{n^2-\sin^2\theta}}U_2 + ik_o h\left(\cos^2\varphi\sin^2\theta - \frac{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}}{\cos\theta + \sqrt{n^2-\sin^2\theta}}\right)U_2\right]\right\}.$$

Now incorporating the results of (4.62), (4.70) and (4.74) we obtain the desired result

$$\Pi_{xo} = \frac{in^2 m\cos\theta\, e^{ik_o h\sqrt{n^2-\sin^2\theta}}}{2\pi\omega\mu_o(n^2\cos\theta + \sqrt{n^2-\sin^2\theta})}\left\{1 + i\kappa\sin\theta\left[\frac{\sin\varphi}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}}\right.\right. \tag{5.2a}$$

$$\left.\left. + \frac{\cos^2\varphi\cos\theta}{2\sin\varphi\sqrt{n^2-\sin^2\theta}(\cos\theta + \sqrt{n^2-\sin^2\theta})} + \frac{ik_o h\cos^2\varphi}{2\sin\varphi}\right]\right\}\frac{e^{ik_o r}}{r}$$

and

$$\Pi_{zo} = \frac{in^2 m\cos\varphi\sin\theta\cos\theta\, e^{ik_o h\sqrt{n^2-\sin^2\theta}}}{2\pi\omega\mu_o(n^2\cos\theta + \sqrt{n^2-\sin^2\theta})}\left\{\frac{\cos\theta - \sqrt{n^2-\sin^2\theta}}{n^2}\right.$$

$$- \frac{i\kappa}{\sin\theta\sin\varphi\sqrt{n^2-\sin^2\theta}}\left[\frac{\sin^2\varphi\sin^2\theta}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}} - \frac{\sin^2\varphi\sin^2\theta + \cos^2\theta}{2(\cos\theta + \sqrt{n^2-\sin^2\theta})}\right. \tag{5.2b}$$

$$\left.\left. + \frac{ik_o h}{2}\left(\cos^2\varphi\sin^2\theta - \frac{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}}{\cos\theta + \sqrt{n^2-\sin^2\theta}}\right)\right]\right\}\frac{e^{ik_o r}}{r}.$$

5.1 b The electric field in spherical coordinates—In Section 2.2 we noted that the electric field was derivable from the Hertzian vector as follows:

$$\vec{E}_o = i\omega\mu_o\vec{\nabla}\times\vec{\Pi}_o.$$

Then each Cartesian component of the electric field is given by

$$\begin{aligned} E_{xo} &= i\omega\mu_o\partial_y\Pi_{zo} \\ E_{yo} &= i\omega\mu_o(\partial_z\Pi_{xo} - \partial_x\Pi_{zo}) \\ E_{zo} &= -i\omega\mu_o\partial_y\Pi_{xo}\quad. \end{aligned} \tag{5.3}$$

To find the components of the electric field in spherical coordinates we use

(3.49) and get

$$E_{\theta o} = i\omega\mu_o\{(\cos\theta\,\sin\varphi\,\partial_x + \sin\theta\,\partial_y)\Pi_{xo} - \cos\theta(\sin\varphi\,\partial_x - \cos\varphi\,\partial_y)\Pi_{zo}\} \quad (5.3a)$$

$$E_{\varphi o} = i\omega\mu_o\{\cos\varphi\,\partial_z\Pi_{xo} - (\cos\varphi\,\partial_x + \sin\varphi\,\partial_y)\Pi_{zo}\} \quad (5.3b)$$

$$E_{ro} = -i\omega\mu_o\{(\cos\theta\,\partial_y - \sin\varphi\sin\theta\,\partial_z)\Pi_{xo} + \sin\theta(\sin\varphi\,\partial_x - \cos\varphi\,\partial_y)\Pi_{zo}\}\,. \quad (5.3c)$$

We now use the results of equation (4.76) for the partial derivatives and obtain for the components of $\vec{E}_o$ in terms of $\vec{\Pi}_o$ as follows:

$$E_{\theta o} = -k_o\omega\mu_o\sin\varphi\,\Pi_{xo} \quad (5.4a)$$

$$E_{\varphi o} = -k_o\omega\mu_o(\cos\varphi\cos\theta\,\Pi_{xo} - \sin\theta\,\Pi_{zo}) \quad (5.4b)$$

$$E_{ro} = 0\,.$$

We note that to the first order in r the electric field is purely transverse as it should be at a far distance from the source (7, p. 67).

5.1 c The magnetic field in spherical coordinates--In Section 2.2 we noted that the magnetic field in the air was derivable from the Hertzian vector as follows:

$$\vec{H}_o = k_o^2\vec{\Pi}_o + \vec{\nabla}(\vec{\nabla}\cdot\vec{\Pi}_o)\,. \quad (5.5)$$

Then the Cartesian components of the vector $\vec{H}_o$ are

$$H_{xo} = k_o^2\Pi_{xo} + \partial_x(\partial_x\Pi_{xo} + \partial_z\Pi_{zo}) \quad (5.6a)$$

$$H_{yo} = \partial_y(\partial_x\Pi_{xo} + \partial_z\Pi_{zo}) \quad (5.6b)$$

$$H_{zo} = k_o^2\Pi_{zo} + \partial_z(\partial_x\Pi_{xo} + \partial_z\Pi_{zo})\,. \quad (5.6c)$$

To find the components of the magnetic field in spherical coordinates we use (3.49) and get

$$\begin{aligned} H_{\theta o} = &\{\cos\varphi\cos\theta(k_o^2 + \partial_x^2) + \partial_x(\sin\varphi\cos\theta\,\partial_y - \sin\theta\,\partial_z)\}\Pi_{xo} \\ &+ \{\cos\theta\,\partial_z(\cos\varphi\,\partial_x + \sin\varphi\,\partial_y) - \sin\theta\,(k_o^2 + \partial_z^2)\}\Pi_{zo} \end{aligned} \quad (5.7a)$$

$$H_{\varphi o} = \left\{ \cos\varphi \, \partial_x \partial_y - \sin\varphi \, (k_o^2 + \partial_x^2) \right\} \Pi_{xo} + \partial_z(\cos\varphi \, \partial_y - \sin\varphi \, \partial_x) \Pi_{zo} \tag{5.7b}$$

$$H_{ro} = \left\{ \cos\varphi \sin\theta \, (k_o^2 + \partial_x^2) + \sin\varphi \sin\theta \, \partial_x \partial_y + \cos\theta \, \partial_x \partial_z \right\} \Pi_{xo} + \left\{ \cos\theta (k_o^2 + \partial_z^2) + \partial_z (\cos\varphi \sin\theta \, \partial_x + \sin\varphi \sin\theta \, \partial_y) \right\} \Pi_{zo} \tag{5.7c}$$

Now we use the results of equation (4.76) for the partial derivatives and obtain for the components of $\vec{H}_o$ in terms of $\vec{\Pi}_o$ as follows:

$$H_{\theta o} = k_o^2 \left[\cos\varphi \cos\theta \, \Pi_{xo} - \sin\theta \, \Pi_{zo} \right] \tag{5.8a}$$

$$H_{\varphi o} = - k_o^2 \sin\varphi \, \Pi_{xo} \tag{5.8b}$$

$$H_{ro} = 0 \tag{5.8c}$$

We note in passing that to the first order the magnetic field is purely transverse as it should be.

We note the following useful relationships among transverse electric and magnetic fields:

$$\frac{E_{\theta o}}{H_{\varphi o}} = - \frac{E_{\varphi o}}{H_{\theta o}} = Z_o \tag{5.9}$$

5.1 d The radiation field—We shall now write the explicit forms of the field components. Using the results of (5.4) as well as those of (5.2a) and (5.2b), we obtain for the components of the radiation field

$$E_{\theta o} = - \frac{i k_o m \cos\theta \, e^{i k_o h \sqrt{n^2 - \sin^2\theta}}}{2\pi (n^2 \cos\theta + \sqrt{n^2 - \sin^2\theta})} \left\{ n^2 \sin\varphi + i\eta \sin\theta \left[\frac{\sin^2\varphi}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}} + \frac{\cos^2\varphi \cos\theta}{2\sqrt{n^2 - \sin^2\theta}(\cos\theta + \sqrt{n^2 - \sin^2\theta})} + \frac{i k_o h \cos^2\varphi}{2} \right\} \frac{e^{i k_o r}}{r} \right. \tag{5.10a}$$

and

$$E_{\varphi o} = -\frac{ik_o m\cos\varphi\cos\theta\, e^{ik_o h\sqrt{n^2-\sin^2\theta}}}{2\pi(n^2\cos\theta+\sqrt{n^2-\sin^2\theta})}\Big\{\sqrt{n^2-\sin^2\theta}\left(\cos\theta\sqrt{n^2-\sin^2\theta}+\sin^2\theta\right)$$

$$+\frac{i\eta\sin\theta\sin\varphi}{\sqrt{n^2-\sin^2\theta}}\left[\frac{\cos\theta\sqrt{n^2-\sin^2\theta}+\sin^2\theta}{n^2\cos\theta+\sqrt{n^2-\sin^2\theta}}-\frac{1}{2(\cos\theta+\sqrt{n^2-\sin^2\theta})}\right. \tag{5.10b}$$

$$\left.-\frac{ik_o h(n^2\cos\theta+\sqrt{n^2-\sin^2\theta})}{2(\cos\theta+\sqrt{n^2-\sin^2\theta})}\right]\Big\}\frac{e^{ik_o r}}{r}.$$

We note that in general the field patterns will not be symmetric in the θ- as well as in the φ-planes.

We note that some of the components of the above field expressions become infinite when $\sin\theta = n$. This is a direct consequence of the approximation we have introduced earlier in equation (3.7) which, as we stated, was not valid for $s = 0$ corresponding to $\sin\theta = n$ in the above results. The results for the radiation field are valid, however, everywhere except in the neighborhood of the point $\sin\theta = n$ and, of course, within the limitations of the approximation itself. This statement we shall prove formally in the latter part of this work when we discuss the solutions to the line sources in Parts II and III.

As a partial check on the above results we shall go to the limit as $n \rightarrow 1$. We should, in this case, obtain the well-known solution for an isolated magnetic dipole. To this end we note immediately from (5.4a) and (5.4b) that when $h = 0$ then

$$\Pi_{xo} = \frac{im}{4\pi\omega\mu_o}\,\frac{e^{ik_o r}}{r}$$
$$\Pi_{zo} = 0 \tag{5.11}$$

which is in agreement with the well-known result (25, p. 184).

5.2 THE POWER FLOW

In the preceding section we found the electric and magnetic field components in the air. The form of these components reveals certain additions that are due to the anisotropy of the plasma. In this section we shall determine the effect of these additions on the direction and magnitude of the energy carried by the electromagnetic wave.

The time-averaged energy density carried by an electromagnetic wave is given by the Poynting vector

$$\vec{S} = \tfrac{1}{2}\,\mathrm{Re}\left\{\vec{E} \times \vec{H}^*\right\} \tag{5.12}$$

which in our case will have only a single component in the radial direction given by

$$S_{\rho o} = \tfrac{1}{2}\,\mathrm{Re}\left\{E_{\theta o} H^*_{\varphi o} - E_{\varphi o} H^*_{\theta o}\right\} \tag{5.13}$$

which in view of the relationships in equation (5.9) can be written

$$S_{\rho o} = \frac{1}{2 Z_o}\left(|E_{\theta o}|^2 + |E_{\varphi o}|^2 \right). \tag{5.14}$$

Now substituting for $E_{\theta o}$ and $E_{\varphi o}$ from (5.10a) and (5.10b) respectively, we obtain

$$S_{\rho o} = \frac{k_o^2\, m^2 \cos^2\theta}{8\pi^2 r^2 Z_o \left|n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}\right|^2}\left\{|F_\theta(\theta,\varphi)|^2 + \cos^2\varphi\,|F_\varphi(\theta,\varphi)|^2\right\} \tag{5.15}$$

where

$$F_\theta(\theta,\varphi) = n^2\sin\varphi + i\eta\sin\theta\left[\frac{\sin^2\varphi}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}} + \frac{\cos^2\varphi\cos\theta}{2\sqrt{n^2-\sin^2\theta}\,(\cos\theta + \sqrt{n^2-\sin^2\theta})} + \frac{ik_o h\cos^2\varphi}{2}\right] \tag{5.16a}$$

and

$$F_\varphi(\theta,\varphi) = \sqrt{n^2-\sin^2\theta}\,(\cos\theta\,\sqrt{n^2-\sin^2\theta} + \sin^2\theta)$$

$$+ \frac{i\gamma\sin\theta\sin\varphi}{\sqrt{n^2-\sin^2\theta}}\left[\frac{\cos\theta\,\sqrt{n^2-\sin^2\theta} + \sin^2\theta}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}} - \frac{1 + ik_oh(n^2\cos\theta + \sqrt{n^2-\sin^2\theta})}{2(\cos\theta + \sqrt{n^2-\sin^2\theta})}\right]. \tag{5.16b}$$

The power pattern represented by equation (5.15) was computed and plotted in Figures 5.2 - 5.4 for the dipole situated at the lower edge of the ionosphere. For the purpose of this example the lower edge of the ionosphere was assumed to be homogeneous and sharply bounded, having an electron density $N = 750$ electrons per cubic centimeter and the earth's magnetic field $H_{DC} = .4$ gauss. Since the cyclotron resonance associated with the assumed earth's magnetic field is 1.12×10^6 cycles per second, to stay within the limits of the validity of the high frequency approximation, the lowest frequency considered was 3×10^6 cycles per second giving $\omega_H/\omega = .373$ which was assumed to be adequate.

Examination of the Figures 5.2 - 5.4 reveals that the correction due to the earth's magnetic field manifests itself most strongly in the plane $\varphi = 0$ and $\varphi = \pi/4$ where φ is azimuthal angle measured from the axis of the dipole. Even here this correction is not, however, very large at these frequencies. It can be seen that for moderate polar angles this correction is completely negligible and it begins to be noticeable for $\theta > 60°$. Furthermore, this correction becomes infinite at the critical angle θ where our high frequency approximation does not hold.

5.3 CLOSURE

In this chapter we have completed the approximate high frequency solution to the problem of a horizontal magnetic dipole in magnetoplasma. In particular we found all of the radiation field components in the air and were able to separate definitely the contributions of the plasma's anisotropy.

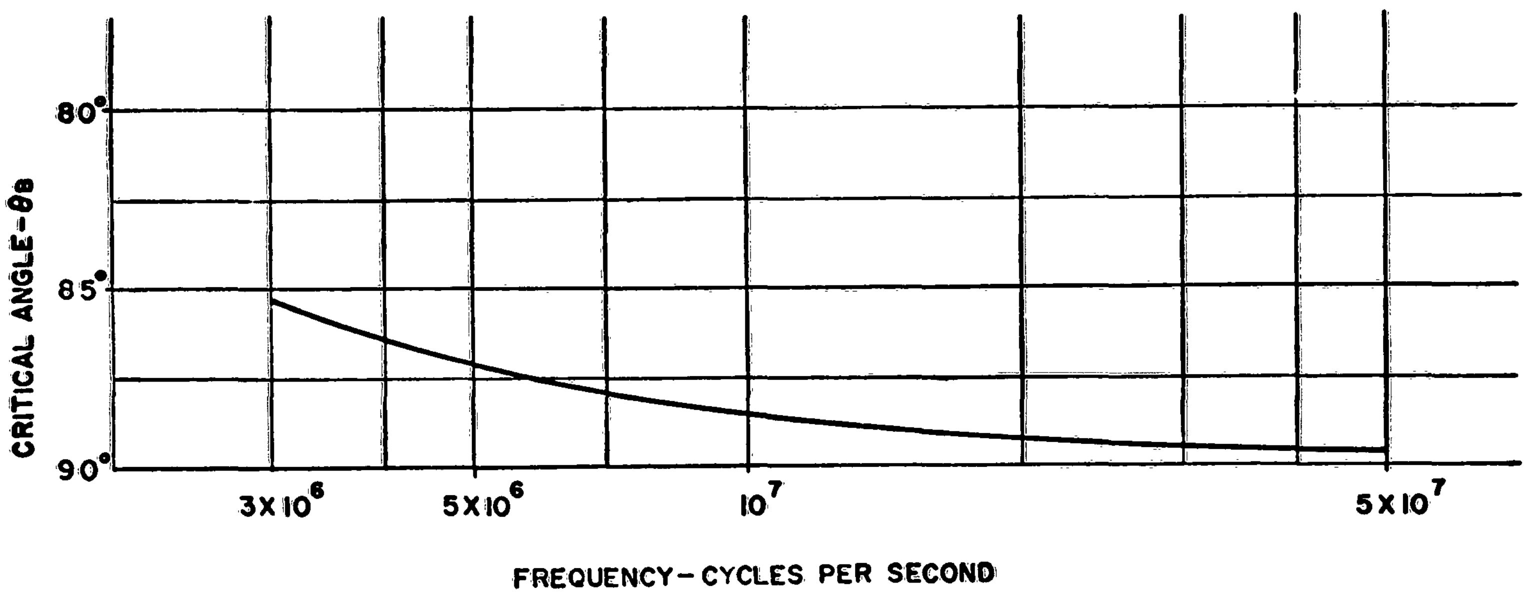

Figure 5.1 - Critical Angle θ_B Versus Frequency for the Dipole Problem

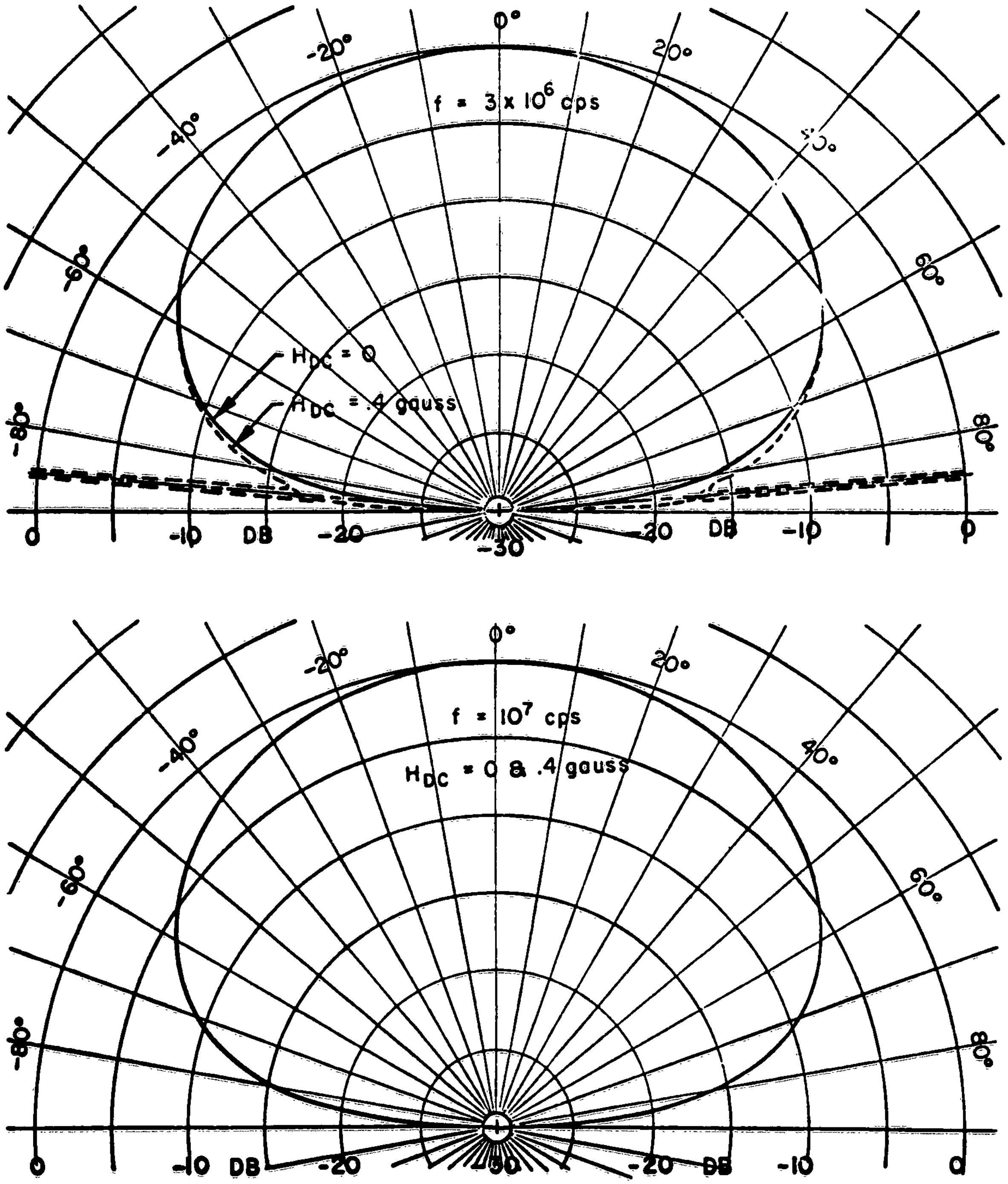

Figure 5.2 - Power Pattern in Air of a Horizontal Magnetic Dipole in Magnetoplasma ($\varphi = 0$, $N = 750$ electrons per cubic centimeter, $h = 1000$ meters, $H_{DC} = 0$ and .4 gauss, $f = 3 \times 10^6$ and 10^7 cycles per second)

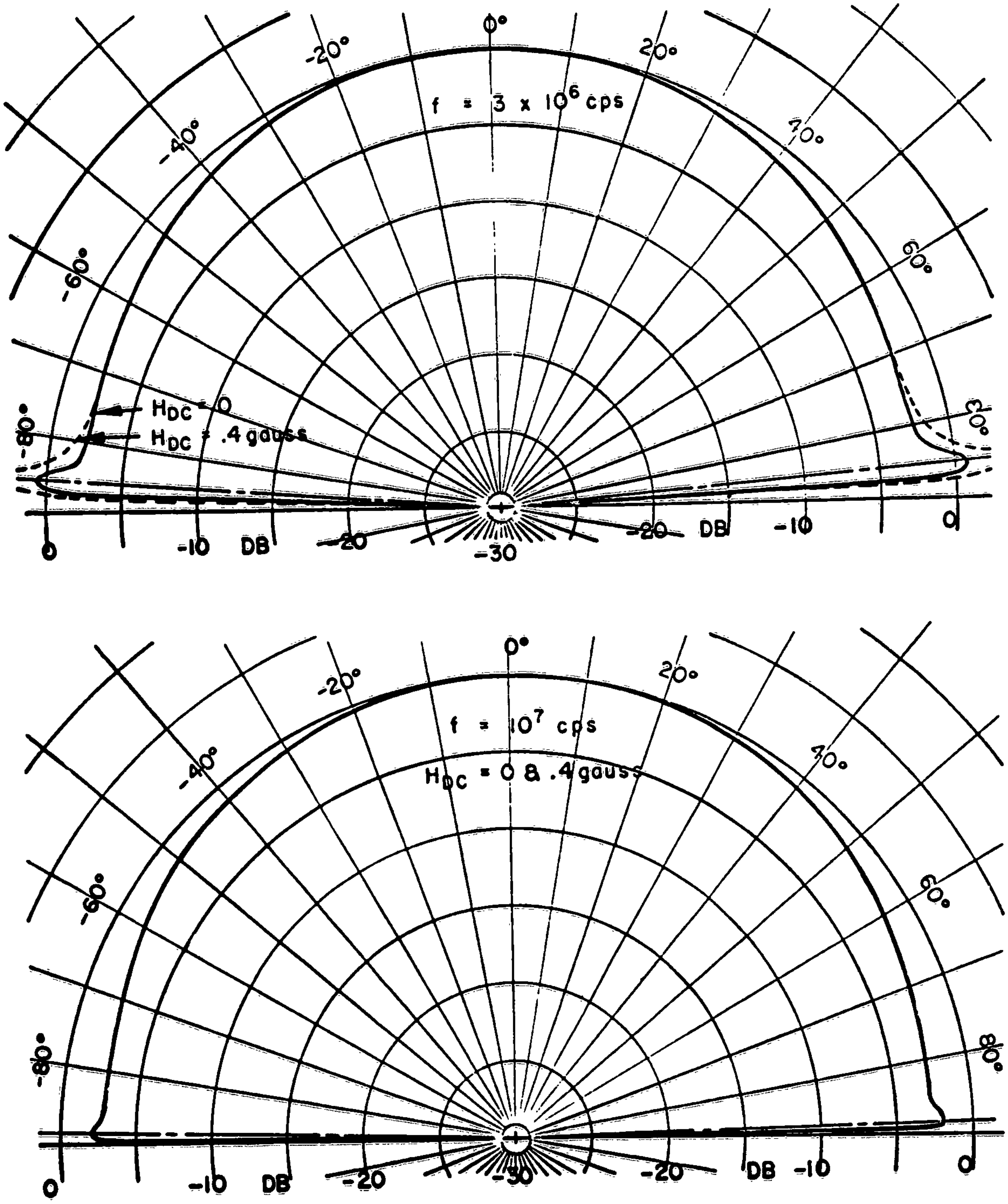

Figure 5.3 - Power Pattern in Air of a Horizontal Magnetic Dipole in Magnetoplasma; $\varphi = \pi/4$, $N = 750$ **electrons per cubic centimeter,** $h = 1000$ **meters,** $H_{DC} = 0$ **and .4 gauss,** $f = 3 \times 10^6$ **and** 10^7 **cycles per second**

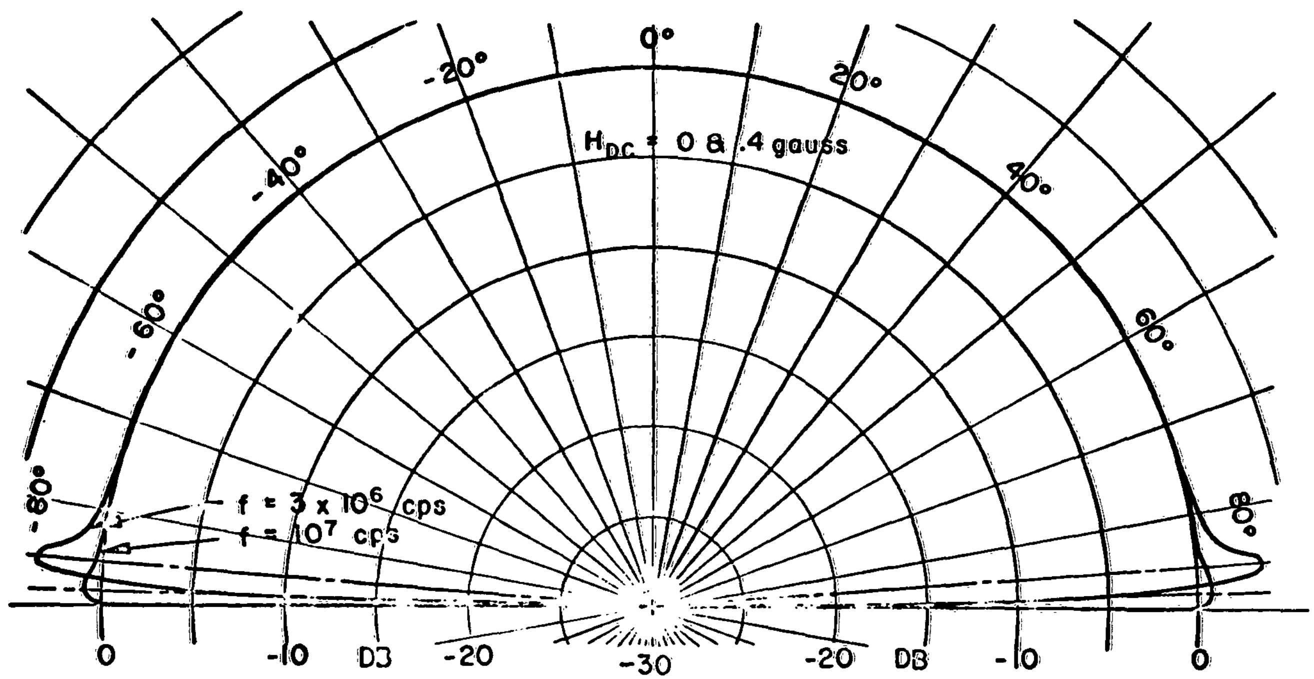

Figure 5.4 - Power Pattern in Air of a Horizontal Magnetic Dipole in Magnetoplasma; $\varphi = \pi/2$, $N = 750$ electrons per cubic centimeter, $H_{DC} = 0$ and $.4$ gauss, $f = 3 \times 10^6$ and 10^7 cycles per second

The results of this analysis were applied to the problem of a horizontal magnetic dipole situated in the lower edge of the ionosphere. The power patterns in the air were obtained for frequencies above 3 megacycles. It appears that in these frequencies, the correction due to the earth's magnetic field manifests itself most strongly in the polar plane through the axis of the dipole and $\pi/4$ from it. Moreover, the patterns are symmetric and the correction due to the earth's magnetic field appears to be significant only for large polar angles, i.e., in the regions close to the interface.

CHAPTER 6

FORMULATION OF THE PROBLEM FOR A MAGNETIC DIPOLE SOURCE IN THE AIR

In this chapter we shall be concerned with the finding of appropriate integral representations for the Cartesian components of the field vectors for the magnetoplasma- and air-half spaces with a source in air. In many respects the present boundary value problem is similar to the one in Chapter 2. We shall, therefore, avail ourselves of the many results of that chapter that are applicable to the present problem.

6.1 STATEMENT OF THE PROBLEM

The geometry of the problem is shown in Fig. 6.1. The horizontal plane, $z = 0$, coincides with the interface between the anisotropic homogeneous plasma and air. Again, we shall call the plasma medium (1) and the air medium (0), and assume that both media have the same magnetic inductive capacity of free space, μ_o. The steady magnetic field H_{DC} shall be oriented as before in the positive x- direction.

6.2 FUNDAMENTAL EQUATIONS

The definition of the present boundary value problem implies solution to Maxwell's equations subject to the usual boundary conditions at the interface and proper behavior at infinity. As before, we shall work with the Hertzian vector of the magnetic type in the air region and with the actual field

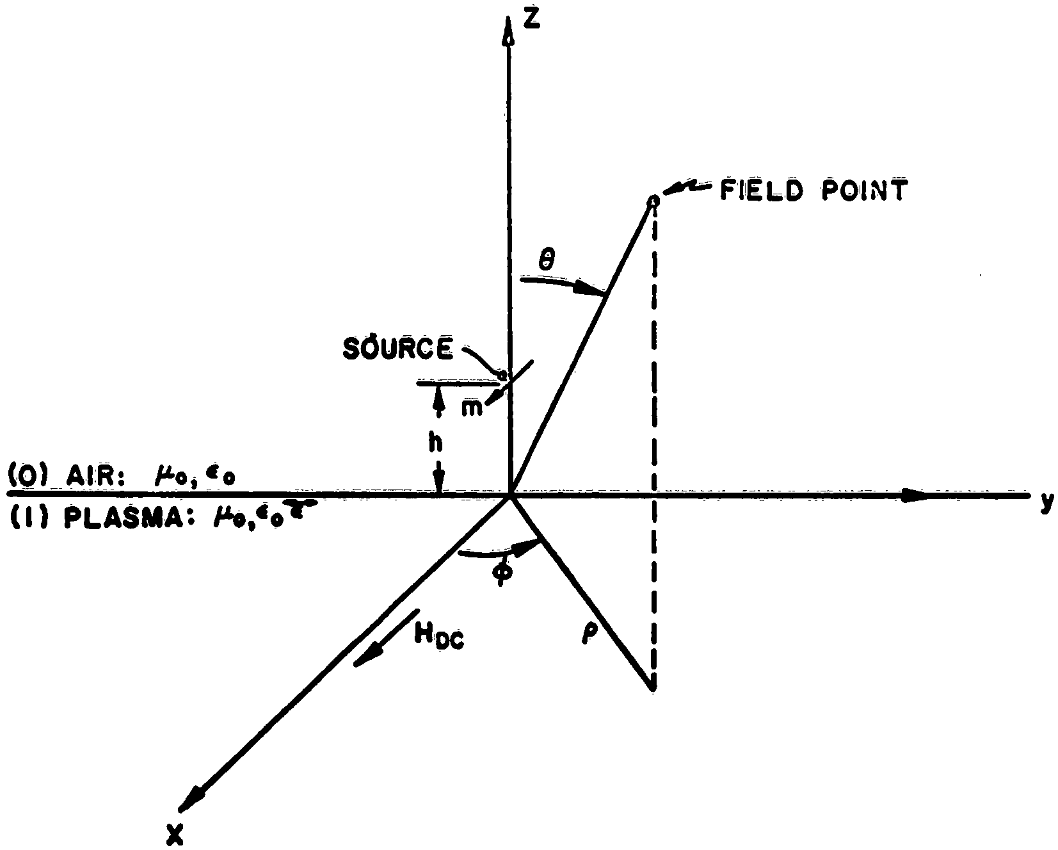

Figure 6.1 - Geometry of the Problem of a Magnetic Dipole in Air

components in the plasma.

6.2 a The field equations--In Section 2.2 it was shown that the source of electromagnetic waves may be regarded as a magnetic dipole singularity which in the present case will be located at a point (0, 0, h).

Then the Hertzian vector in the air will satisfy the inhomogeneous vector wave equation

$$(\nabla^2 + k_0^2)\vec{\Pi}_0 = -\frac{i\,m}{\omega\mu_0}\,\delta(x)\,\delta(y)\,\delta(z-h) \tag{6.1}$$

where

$$\vec{E}_0 = i\omega\mu_0 \vec{\nabla}\times\vec{\Pi}_0 \tag{6.2a}$$

$$\vec{H}_0 = k_0^2\vec{\Pi}_0 + \vec{\nabla}\cdot(\vec{\nabla}\cdot\vec{\Pi}_0)\,. \tag{6.2b}$$

In the plasma the magnetic field must satisfy the homogeneous system of equations of (2.26) which we rewrite here for convenience

$$\begin{bmatrix} \chi k_0^2 + \partial_y^2 + \partial_z^2 & -\partial_x(\partial_y + i\kappa\partial_z) & -\partial_x(-i\kappa\partial_y + \partial_z) \\ -\partial_x(\partial_y - i\kappa\partial_z) & \chi k_0^2 + \partial_x^2 + \frac{\chi}{\zeta}\partial_z^2 & -\frac{\chi}{\zeta}\partial_y\partial_z - i\kappa\partial_x^2 \\ -\partial_x(i\kappa\partial_y + \partial_z) & -\frac{\chi}{\zeta}\partial_y\partial_z + i\kappa\partial_x^2 & \chi k_0^2 + \partial_x^2 + \frac{\chi}{\zeta}\partial_y^2 \end{bmatrix} \begin{bmatrix} H_{x1} \\ H_{y1} \\ H_{z1} \end{bmatrix} = 0 \tag{6.3}$$

and the electric field is given in terms of the magnetic field as follows:

$$\begin{bmatrix} E_{x1} \\ E_{y1} \\ E_{z1} \end{bmatrix} = \frac{1}{i\omega\epsilon_0\chi} \begin{bmatrix} 0 & \frac{\chi}{\zeta}\partial_z & -\frac{\chi}{\zeta}\partial_y \\ -i\kappa\partial_y - \partial_z & i\kappa\partial_x & \partial_x \\ \partial_y - i\kappa\partial_z & -\partial_x & i\kappa\partial_x \end{bmatrix} \begin{bmatrix} H_{x1} \\ H_{y1} \\ H_{z1} \end{bmatrix} \tag{6.4}$$

6.3 FOURIER INTEGRAL REPRESENTATION IN CARTESIAN COORDINATES

As before, we shall employ here the triple Fourier transform pair to simplify the formulation of the present problem.

6.3 a The particular integral corresponding to the source—Applying the results of Section 2.2 to the wave equation of (6.1) we readily obtain

$$\bar{\bar{\Pi}}_{xo}^{(p)} = \frac{-i\,m}{(2\pi)^{3/2}\,\omega\mu_o}\left(\frac{e^{-i\alpha_3 h}}{s_o^2 - \alpha_3^2}\right) \tag{6.5}$$

where $s_o^2 = k_o^2 - \alpha_1^2 - \alpha_2^2$ as before. We invert with respect to the α_3 transform variable to obtain

$$\tilde{\Pi}_{xo}^{(p)} = \frac{-m}{4\pi\,\omega\mu_o}\cdot\frac{e^{is_o|z-h|}}{s_o} \tag{6.6}$$

Now inverting with respect to the α_1 and α_2 transform variables, we obtain the well-known result (2, p. 22)

$$\Pi_{xo}^{(p)} = \frac{-m}{8\pi^2\omega\mu_o}\iint_{-\infty}^{\infty}\frac{e^{i(\alpha_1 x + \alpha_2 y + s_o|z-h|)}}{s_o}\,d\alpha_1\,d\alpha_2 \tag{6.7}$$

6.3 b The complementary integral in the air—To satisfy the boundary conditions at the interface we shall need appropriate complementary solutions of the differential equation (6.1). These can be written at once as follows:

$$\Pi_{xo}^{(c)} = \frac{-m}{8\pi^2\omega\mu_o}\iint_{-\infty}^{\infty}B_1\,e^{i(\alpha_1 x + \alpha_2 y + s_o z)}\,d\alpha_1\,d\alpha_2 \tag{6.8a}$$

and

$$\Pi_{zo}^{(c)} = \frac{-m}{8\pi^2\omega\mu_o}\iint_{-\infty}^{\infty}B_2\,e^{i(\alpha_1 x + \alpha_2 y + s_o z)}\,d\alpha_1\,d\alpha_2 \tag{6.8b}$$

In all of the above integrals we shall require that $\mathrm{Im}\{s_o\} \geqslant 0$ as before.

6.3 c The fields in the magnetoplasma—The fields in the magnetoplasma must satisfy the system of the differential equations in (6.3). Their forms will be identical to those obtained in the previous problem in equations

(2.64a), (2.64b, and (2.64c). Thus, we write

$$H_{x1} = \frac{-mk_0^2}{8\pi^2\omega\mu_0}\int\!\!\int_{-\infty}^{\infty}(A_1 e^{-is_1 z} + A_2 e^{-is_2 z})e^{i(\alpha_1 x+\alpha_2 y)}d\alpha_1 d\alpha_2 \tag{6.9a}$$

$$H_{y1} = \frac{-imk_0^2}{8\pi^2\omega\mu_0}\partial_x\int\!\!\int_{-\infty}^{\infty}\Big\{\frac{\alpha_2\Phi_2(s_1)+i\kappa\chi\zeta k_0^2 s_1}{\Phi_1(s_1)}A_1 e^{-is_1 z} + \frac{\alpha_2\Phi_2(s_2)+i\kappa\chi\zeta k_0^2 s_2}{\Phi_1(s_2)}A_2 e^{-is_2 z}\Big\}e^{i(\alpha_1 x+\alpha_2 y)}d\alpha_1 d\alpha_2 \tag{6.9b}$$

$$H_{z1} = \frac{-imk_0^2}{8\pi^2\omega\mu_0}\partial_x\int\!\!\int_{-\infty}^{\infty}\Big\{\frac{-s_1\Phi_2(s_1)+i\kappa\chi\zeta k_0^2\alpha_2}{\Phi_1(s_1)}A_1 e^{-is_1 z} + \frac{-s_2\Phi_2(s_2)+i\kappa\chi\zeta k_0^2\alpha_2}{\Phi_1(s_2)}A_2 e^{-is_2 z}\Big\}e^{i(\alpha_1 x+\alpha_2 y)}d\alpha_1 d\alpha_2 \tag{6.9c}$$

where, as before

$$s_{1,2}^2 = \frac{1}{2}\Big\{(\chi+\zeta)k_0^2 - \frac{\chi+\zeta(1-\kappa^2)}{\chi}\alpha_1^2 - 2\alpha_2^2 \pm\sqrt{\Big[(\chi-\zeta)k_0^2 - \frac{\chi-\zeta(1-\kappa^2)}{\chi}\alpha_1^2\Big]^2 + 4\zeta\kappa^2 k_0^2\alpha_1^2}\Big\} \tag{6.10a}$$

$$\Phi_1(s_{1,2}) = (\chi k_0^2-\alpha_1^2)\big[\zeta(\chi k_0^2-\alpha_1^2)-\chi(\alpha_2^2+s_{1,2}^2)\big]-\kappa^2\zeta\alpha_1^4 \tag{6.10b}$$

$$\Phi_2(s_{1,2}) = \zeta\chi k_0^2 - \zeta(1-\kappa^2)\alpha_1^2 - \chi(\alpha_2^2+s_{1,2}^2)\,. \tag{6.10c}$$

6.4 THE BOUNDARY CONDITIONS

The boundary conditions to be satisfied in the present problem are identical to those of Chapter 2. However, due to the fact that the source of the electromagnetic waves is now situated in the air rather than in the

magnetoplasma, they can be restated in more convenient form.

6.4 a Statement of the boundary conditions—The boundary conditions to be satisfied by the Cartesian components of the field vectors require continuity of the tangential components of the electric and magnetic fields at the interface $z = 0$. This was formally stated in equation (2.70). For the purpose of the present problem we put these equations in the following form:

$$k_o^2 \check{\Pi}_{xo} = \check{H}_{x1} - \frac{\alpha_1}{\alpha_2} \check{H}_{y1} \tag{6.11a}$$

$$k_o^2 \partial_z \check{\Pi}_{zo} = -i \alpha_1 \check{H}_{x1} - \frac{i(k_o^2 - \alpha_1^2)}{\alpha_2} \check{H}_{y1} \tag{6.11b}$$

$$k_o^2 \check{\Pi}_{zo} = \frac{1}{\zeta} \check{H}_{z1} + \frac{i}{\alpha_2 \zeta} \partial_z \check{H}_{y1} \tag{6.11c}$$

$$k_o^2 \partial_z \Pi_{xo} = \frac{1}{\chi} \partial_z \check{H}_{x1} - \frac{\alpha_1}{\zeta \alpha_2} \partial_z \check{H}_{y1} - \frac{\kappa \alpha_2}{\chi} \check{H}_{x1} + \frac{\kappa \alpha_1}{\chi} \check{H}_{y1} + \frac{i\alpha_1(\chi - \zeta)}{\zeta \chi} \check{H}_{z1} \tag{6.11d}$$

where we used $\vee$ to denote the field expressions with the integral signs removed.

Carrying out indicated mathematical operations, one obtains four algebraic equations with four unknowns which we shall write in a symbolic form as follows:

$$\begin{bmatrix} c_{11} & 0 & c_{13} & c_{14} \\ 0 & c_{22} & c_{23} & c_{24} \\ 0 & c_{32} & c_{33} & c_{34} \\ c_{41} & 0 & c_{43} & c_{44} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ 0 \\ b_2 \end{bmatrix} \tag{6.12}$$

where

$$c_{11} = 1 \tag{6.13a}$$

$$c_{13} = -\frac{\alpha_2 \left[\Phi_1(s_1) + \alpha_1^2 \Phi_2(s_1)\right] + i\kappa\chi\zeta k_o^2 s_1 \alpha_1^2}{\alpha_2 \Phi_1(s_1)} \tag{6.13b}$$

$$c_{14} = -\frac{\alpha_2 \left[\Phi_1(s_2) + \alpha_1^2 \Phi_2(s_2)\right] + i\kappa\chi\zeta k_o^2 s_2 \alpha_1^2}{\alpha_2 \Phi_1(s_2)} \tag{6.13c}$$

$$c_{22} = \frac{s_0}{\alpha_1} \tag{6.13d}$$

$$c_{23} = \frac{\alpha_2\left[\Phi_1(s_1) - (k_0^2-\alpha_1^2)\Phi_2(s_1)\right] - i\kappa\chi\zeta k_0^2(k_0^2-\alpha_1^2)s_1}{\alpha_2\Phi_1(s_1)} \tag{6.13e}$$

$$c_{24} = \frac{\alpha_2\left[\Phi_1(s_2) - (k_0^2-\alpha_1^2)\Phi_2(s_2)\right] - i\kappa\chi\zeta k_0^2(k_0^2-\alpha_1^2)s_2}{\alpha_2\Phi_1(s_2)} \tag{6.13f}$$

$$c_{32} = \frac{\zeta}{\alpha_1} \tag{6.13g}$$

$$c_{33} = \frac{i\kappa\chi\zeta k_0^2(s_1^2+\alpha_2^2)}{\alpha_2\Phi_1(s_1)} \tag{6.13h}$$

$$c_{34} = \frac{i\kappa\chi\zeta k_0^2(s_2^2+\alpha_2^2)}{\alpha_2\Phi_1(s_2)} \tag{6.13i}$$

$$c_{41} = 1 \tag{6.13j}$$

$$c_{43} = \frac{\zeta\alpha_2\Phi_1(s_1)(s_1-i\kappa\alpha_2) + \alpha_1^2(\chi s_1 - i\kappa\alpha_2\zeta)\left[\alpha_2\Phi_2(s_1) + i\kappa\chi\zeta k_0^2 s_1\right]}{s_0\alpha_2\zeta\chi\Phi_1(s_1)} + \frac{\alpha_1^2(\chi-\zeta)\left[-s_1\Phi_2(s_1) + i\kappa\chi\zeta k_0^2\alpha_2\right]}{\zeta\chi\Phi_1(s_1)} \tag{6.13k}$$

$$c_{44} = \frac{\zeta\alpha_2\Phi_1(s_2)(s_2-i\kappa\alpha_2) + \alpha_1^2(\chi s_2 - i\kappa\alpha_2\zeta)\left[\alpha_2\Phi_2(s_2) + i\kappa\chi\zeta k_0^2 s_2\right]}{s_0\alpha_2\zeta\chi\Phi_1(s_2)} + \frac{\alpha_1^2(\chi-\zeta)\left[-s_2\Phi_2(s_2) + i\kappa\chi\zeta k_0^2\alpha_2\right]}{\zeta\chi\Phi_1(s_2)} \tag{6.13l}$$

$$b_1 = -\frac{e^{is_0h}}{s_0} \tag{6.13m}$$

$$b_2 = \frac{e^{is_0h}}{s_0}. \tag{6.13n}$$

6.5 HIGH FREQUENCY APPROXIMATION

The determination of the unknowns B_1, B_2, A_1, and A_2 from the system of equations (6.12) is a straight-forward but tedious process. The results would necessarily be very lengthy and probably not too useful. Therefore, as in the case of the dipole in the magnetoplasma we shall forego their evaluation in

this form. Instead, as before, we introduce the high frequency approximation (3.7) which we rewrite here for convenience

$$s_{1,2} = s \pm \frac{\kappa k_1 \alpha_1}{2\,s} \tag{6.14}$$

where $s = (k_1^2 - \alpha_1^2 - \alpha_2^2)^{\frac{1}{2}}$ as before.

<u>6.5 a Evaluation of the boundary coefficients</u>—Using the high frequency approximation we find the approximate value of the matrix coefficients a_{ij} as follows:

$$a_{11} = 1 \tag{6.15a}$$

$$a_{13} = -k_1\left\{\frac{(\alpha_2 k_1 - i\alpha_1 s)(s^2+\alpha_2^2) + \kappa\alpha_1^2\left[(s^2-\alpha_2^2)^{ik_1/2s} - \alpha_1\alpha_2\right]}{\alpha_2(s^2+\alpha_2^2)^2}\right\} \tag{6.15b}$$

$$a_{14} = -k_1\left\{\frac{(\alpha_2 k_1 + i\alpha_1 s)(s^2+\alpha_2^2) + \kappa\alpha_1^2\left[(s^2-\alpha_2^2)^{ik_1/2s} + \alpha_1\alpha_2\right]}{\alpha_2(s^2+\alpha_2^2)^2}\right\} \tag{6.15c}$$

$$a_{22} = \frac{s_0}{\alpha_1} \tag{6.15d}$$

$$a_{23} = \frac{\left[ik_1 s(s_0^2+\alpha_2^2) + k_0^2\alpha_1\alpha_2(n^2-1)\right](s^2+\alpha_2^2) + \kappa k_1\alpha_1(s_0^2+\alpha_2^2)\left[\alpha_1\alpha_2 - (s^2-\alpha_2^2)^{ik_1/2s}\right]}{\alpha_1\alpha_2(s^2+\alpha_2^2)^2} \tag{6.15e}$$

$$a_{24} = -\frac{\left[ik_1 s(s_0^2+\alpha_2^2) - k_0^2\alpha_1\alpha_2(n^2-1)\right](s^2+\alpha_2^2) + \kappa k_1\alpha_1(s_0^2+\alpha_2^2)\left[\alpha_1\alpha_2 - (s^2-\alpha_2^2)^{ik_1/2s}\right]}{\alpha_1\alpha_2(s^2+\alpha_2^2)^2} \tag{6.15f}$$

$$a_{32} = \frac{n^2}{\alpha_1} \tag{6.15g}$$

$$a_{33} = -\frac{i\,k_1}{\alpha_1\alpha_2} \tag{6.15h}$$

$$a_{34} = \frac{i\,k_1}{\alpha_1\alpha_2} \tag{6.15i}$$

$$a_{41} = 1 \tag{6.15j}$$

$$a_{43} = k_1\left\{\frac{s(s^2+\alpha_2^2)(\alpha_2 k_1 - i\alpha_1 s) - \kappa k_1^2\alpha_2\left[(s^2-\alpha_2^2)^{\alpha_1/2s} + ik_1\alpha_2\right]}{n^2\alpha_2 s_0(s^2+\alpha_2^2)^2}\right\} \tag{6.15k}$$

$$a_{44} = k_1\left\{\frac{s(s^2+\alpha_2^2)(\alpha_2 k_1 + i\alpha_1 s) + \kappa k_1^2\alpha_2\left[(s^2-\alpha_2^2)^{\alpha_1/2s} - ik_1\alpha_2\right]}{n^2\alpha_2 s_0(s^2+\alpha_2^2)^2}\right\} \tag{6.15l}$$

Furthermore, we can reduce the system of equations in (6.12) to a simpler system as follows:

$$\begin{bmatrix} c_{11} & 0 & c_{13} & c_{14} \\ 0 & c_{22} & c_{23} & c_{24} \\ 0 & 0 & d_{33} & d_{34} \\ 0 & 0 & d_{43} & d_{44} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ 0 \\ -2b_1 \end{bmatrix} \tag{6.16}$$

where

$$d_{33} = a_{21} \tag{6.17a}$$

$$d_{34} = a_{22} \tag{6.17b}$$

$$d_{43} = \frac{a_{11}}{n^2 s_o \alpha_2} \tag{6.17c}$$

$$d_{44} = \frac{a_{12}}{n^2 s_o \alpha_2} \tag{6.17d}$$

and a_{11}, a_{12}, a_{21}, and a_{22} are given by (3.11).

Using the above results we obtain finally

$$B_1 = \left\{ -\frac{1}{s_o} + \frac{2n^2}{s+n^2 s_o} + i\eta\left[\frac{2\alpha_2}{(s+n^2 s_o)^2} + \frac{\alpha_1^2 s_o}{\alpha_2 s(s+s_o)(s+n^2 s_o)}\right]\right\} e^{is_o h} \tag{6.18a}$$

$$B_2 = -\left\{\frac{2(n^2-1)\alpha_1}{(s+s_o)(s+n^2 s_o)} + i\eta\left[\frac{2\alpha_1\alpha_2}{s(s+n^2 s_o)} - \frac{\alpha_1(s_o^2+\alpha_2^2)}{\alpha_2 s(s+s_o)(s+n^2 s_o)}\right]\right\} e^{is_o h}. \tag{6.18b}$$

<u>6.5 b Integral representation of the Cartesian components of the Hertzian vector</u>—Substituting the results of (6.18a) and (6.18b) into (6.8a) and (6.8b) respectively, one obtains

$$\begin{aligned} \Pi_{xo} = \frac{-m}{8\pi^2\omega\mu_o}\iint_{-\infty}^{\infty}\Bigg\{ & \frac{e^{is_o|z-h|}}{s_o} - \frac{e^{is_o(z+h)}}{s_o} + \frac{2n^2 e^{is_o(z+h)}}{s+n^2 s_o} \\ & + i\eta\left[\frac{2\alpha_2}{(s+n^2 s_o)^2} + \frac{\alpha_1^2 s_o}{\alpha_2 s(s+s_o)(s+n^2 s_o)}\right] e^{is_o(z+h)}\Bigg\} e^{i(\alpha_1 x+\alpha_2 y)}\, d\alpha_1\, d\alpha_2 \end{aligned} \tag{6.19a}$$

$$\Pi_{zo} = \frac{m}{8\pi^2\omega\mu_o}\iint_{-\infty}^{\infty}\left\{\frac{2(n^2-1)\alpha_1}{(s+s_o)(s+n^2s_o)} + i\eta\left[\frac{2\alpha_1\alpha_2}{s(s+n^2s_o)^2} - \frac{\alpha_1(s_o^2+\alpha_2^2)}{\alpha_2 s(s+s_o)(s+n^2s_o)}\right]\right\}$$

$$\cdot e^{i[\alpha_1 x+\alpha_2 y+s_o(z+h)]}d\alpha_1 d\alpha_2 . \quad (6.19b)$$

We note that when h = 0, (6.19a) and (6.19b) are exactly the same as (3.22a) and (3.22b) respectively as they should be.

Using the relationships of equation (3.23b) we recast the above integrals in more convenient forms

$$\Pi_{xo} = \frac{-m}{8\pi^2\omega\mu_o}\left\{\iint_{-\infty}^{\infty}\left[\frac{e^{is_o|z-h|}-e^{is_o(z+h)}}{s_o}\right]e^{i(\alpha_1 x+\alpha_2 y)}d\alpha_1 d\alpha_2\right.$$

$$+\eta\left[2\partial_y\iint_{-\infty}^{\infty}\frac{e^{i[\alpha_1 x+\alpha_2 y+s_o(z+h)]}}{(s+n^2s_o)^2}d\alpha_1 d\alpha_2\right. \quad (6.20a)$$

$$\left.-\partial_x^2\partial_z\iint_{-\infty}^{\infty}\frac{e^{i[\alpha_1 x+\alpha_2 y+s_o(z+h)]}}{\alpha_2 s(s+s_o)(s+n^2s_o)}d\alpha_1 d\alpha_2\quad\right\}$$

and

$$\Pi_{zo} = \frac{im}{8\pi^2\omega\mu_o}\partial_x\left\{2(1-n^2)\iint_{-\infty}^{\infty}\frac{e^{i[\alpha_1 x+\alpha_2 y+s_o(z+h)]}}{(s+s_o)(s+n^2s_o)}d\alpha_1 d\alpha_2\right.$$

$$-\eta\left[2\partial_y\iint_{-\infty}^{\infty}\frac{e^{i[\alpha_1 x+\alpha_2 y+s_o(z+h)]}}{s(s+n^2s_o)^2}d\alpha_1 d\alpha_2\right. \quad (6.20b)$$

$$\left.\left.+i(\partial_y^2+\partial_z^2)\iint_{-\infty}^{\infty}\frac{e^{i[\alpha_1 x+\alpha_2 y+s_o(z+h)]}}{\alpha_2 s(s+s_o)(s+n^2s_o)}d\alpha_1 d\alpha_2\right]\right\}.$$

<u>6.5 c Definition of fundamental integrals</u>—To facilitate the evaluation of the above definite integrals we shall, as in Section 3.3, define certain fundamental integrals from which all others could be derived by differentiation. To this end we define

$$V_1 = \frac{1}{k_0\pi}\iint_{-\infty}^{\infty} \frac{e^{i[\alpha_1 x+\alpha_2 y+s_0(z+h)]}}{s+n^2 s_0}\, d\alpha_1 d\alpha_2 \tag{6.21a}$$

$$V_2 = \frac{1}{\pi}\iint_{-\infty}^{\infty} \frac{e^{i[\alpha_1 x+\alpha_2 y+s_0(z+h)]}}{(s+n^2 s_0)^2}\, d\alpha_1 d\alpha_2 \tag{6.21b}$$

$$V_3 = \frac{k_0}{\pi}\iint_{-\infty}^{\infty} \frac{e^{i[\alpha_1 x+\alpha_2 y+s_0(z+h)]}}{s(s+n^2 s_0)^2}\, d\alpha_1 d\alpha_2 \tag{6.21c}$$

$$V_4 = \frac{1}{\pi}\iint_{-\infty}^{\infty} \frac{e^{i[\alpha_1 x+\alpha_2 y+s_0(z+h)]}}{(s+s_0)(s+n^2 s_0)}\, d\alpha_1 d\alpha_2 \tag{6.21d}$$

$$V_5 = \frac{k_0}{2\pi i}\iint_{-\infty}^{\infty} \frac{e^{i[\alpha_1 x+\alpha_2 y+s_0(z+h)]}}{\alpha_2 s(s+s_0)(s+n^2 s_0)}\, d\alpha_1 d\alpha_2\ . \tag{6.21e}$$

Employing the above definitions we can rewrite the Cartesian components of the Hertzian vector as follows:

$$\Pi_{x0} = \frac{-m}{8\pi\omega\mu_0}\left\{\frac{1}{\pi}\iint_{-\infty}^{\infty}\left[\frac{e^{is_0|z-h|}-e^{is_0(z+h)}}{s_0}\right]e^{i(\alpha_1 x+\alpha_2 y)}d\alpha_1 d\alpha_2 \right.$$
$$\left. + 2k_0 n^2 V_1 + \eta\left[2\partial_y V_2 - \frac{2i}{k_0^2}\partial_x^2\partial_z V_5\right]\right\} \tag{6.22a}$$

$$\Pi_{z0} = \frac{im}{8\pi\omega\mu_0}\partial_x\left\{2(1-n^2)V_4 - \eta\left[\frac{2}{k_0}\partial_y V_3 - \frac{2}{k_0^2}(\partial_y^2+\partial_z^2)V_5\right]\right\}. \tag{6.22b}$$

<u>6.5 d Transformation to cylindrical coordinates in configuration and transform spaces</u>—Employing the transformations of equations (3.28) and (3.29), we rewrite the fundamental integrals as follows:

$$V_1 = \frac{1}{k_o}\int_{-\infty}^{\infty} \frac{e^{is_o(z+h)}}{s + n^2 s_o} H_o^{(1)}(\lambda\rho)\lambda\, d\lambda \tag{6.23a}$$

$$V_2 = \int_{-\infty}^{\infty} \frac{e^{is_o(z+h)}}{(s + n^2 s_o)^2} H_o^{(1)}(\lambda\rho)\lambda\, d\lambda \tag{6.23b}$$

$$V_3 = k_o \int_{-\infty}^{\infty} \frac{e^{is_o(z+h)}}{s(s + n^2 s_o)^2} H_o^{(1)}(\lambda\rho)\lambda\, d\lambda \tag{6.23c}$$

$$V_4 = \int_{-\infty}^{\infty} \frac{e^{is_o(z+h)}}{(s+s_o)(s+n^2 s_o)} H_o^{(1)}(\lambda\rho)\lambda\, d\lambda \tag{6.23d}$$

$$V_5 = k_o \sum_{\nu=0}^{\infty} (-1)^{\nu} \sin(2\nu+1)\varphi \int_{-\infty}^{\infty} \frac{e^{is_o(z+h)}}{s(s+s_o)(s+n^2 s_o)} H_{2\nu+1}^{(1)}(\lambda\rho)\, d\lambda . \tag{6.23e}$$

<u>6.5 e Transformation to spherical coordinates in configuration and transform spaces</u>—In the configuration space we transform as shown in Fig. 6.1. We note

$$\begin{aligned} z - h &= r_o \cos\theta_o \\ \rho &= r_o \sin\theta_o \\ r_o &= \sqrt{\rho^2 + (z-h)^2} \end{aligned} \tag{6.24}$$

and

$$\begin{aligned} z + h &= r \cos\theta \\ \rho &= r \sin\theta \\ r &= \sqrt{\rho^2 + (z+h)^2} . \end{aligned} \tag{6.25}$$

In the transform space we put as before

$$\lambda = k_o \sin\beta \tag{6.26}$$

and as a consequence obtain the following results:

$$V_1 = \int_{\Gamma_1} \frac{\sin\beta\cos\beta\, \hat{H}_o^{(1)}(k_o\rho\sin\beta)}{n^2\cos\beta + \sqrt{n^2-\sin^2\beta}}\, e^{ik_o r\cos(\beta-\theta)}\, d\beta \tag{6.27a}$$

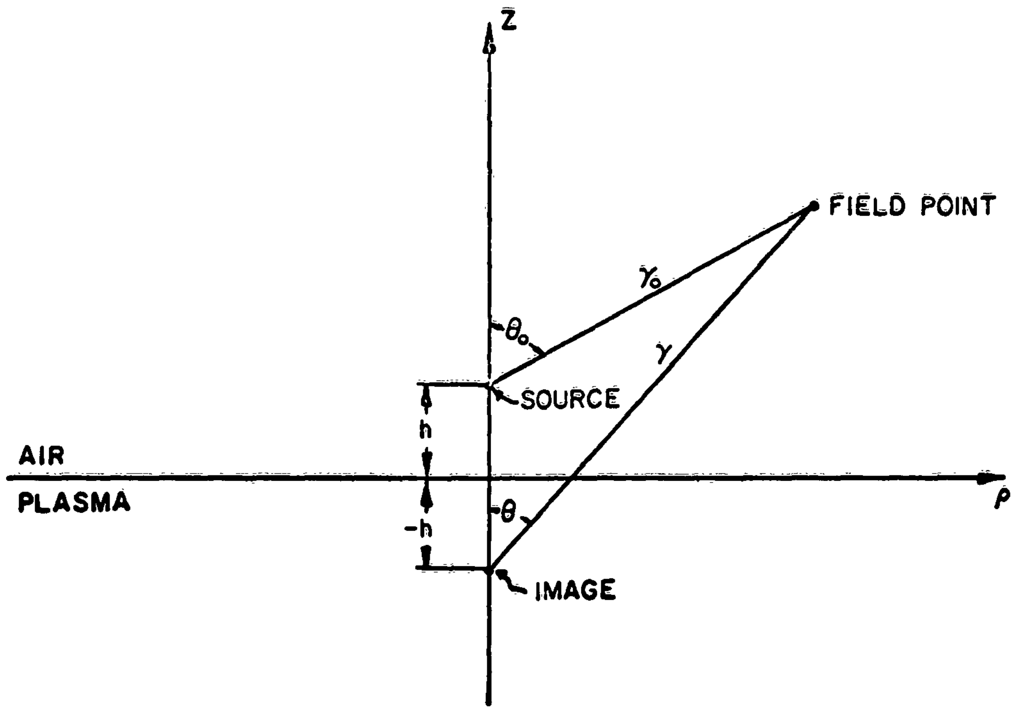

Figure 6.2 - Geometry of the Transformation in the Configuration Space

$$V_2 = \int_{\Gamma_1} \frac{\sin\beta\cos\beta\,\hat{H}_0^{(1)}(k_0\rho\sin\beta)}{(n^2\cos\beta+\sqrt{n^2-\sin^2\beta}\,)^2}\,e^{ik_0r\cos(\beta-\theta)}d\beta \tag{6.27b}$$

$$V_3 = \int_{\Gamma_1} \frac{\sin\beta\cos\beta\,\hat{H}_0^{(1)}(k_0\rho\sin\beta)}{\sqrt{n^2-\sin^2\beta}\,(n^2\cos\beta+\sqrt{n^2-\sin^2\beta}\,)^2}\,e^{ik_0r\cos(\beta-\theta)}d\beta \tag{6.27c}$$

$$V_4 = \int_{\Gamma_1} \frac{\sin\beta\cos\beta\,\hat{H}_0^{(1)}(k_0\rho\sin\beta)}{(\cos\beta+\sqrt{n^2-\sin^2\beta}\,)(n^2\cos\beta+\sqrt{n^2-\sin^2\beta}\,)}\,e^{ik_0r\cos(\beta-\theta)}d\beta \tag{6.27d}$$

$$V_5 = \sum_{\nu=0}^{\infty}(-1)^\nu\sin(2\nu+1)\varphi\int_{\Gamma_1} \frac{\cos\beta\,\hat{H}_{2\nu+1}^{(1)}(k_0\rho\sin\beta)\,e^{ik_0r\cos(\beta-\theta)}}{\sqrt{n^2-\sin^2\beta}\,(\cos\beta+\sqrt{n^2-\sin^2\beta}\,)(n^2\cos\beta+\sqrt{n^2-\sin^2\beta}\,)}\,d\beta. \tag{6.27e}$$

6.6 CLOSURE

In the foregoing chapter we first rigorously formulated the problem of a horizontal magnetic dipole in the air in the presence of a magnetoplasma half-space. Following the rigorous formulation we then applied the high frequency approximation as in Chapter 3. This enabled us to find the explicit form of the boundary coefficients and the approximate integral representations of the Cartesian components of the Hertzian vector.

Furthermore, as a partial check on the results of this chapter as well as those of Chapter 3, we have shown that the integral representations of the Hertzian vector in the cases of the source in the magnetoplasma and in the air are identical when the depth of the source burial is zero, that is, when the source is situated right on the boundary.

CHAPTER 7

FIELDS AND POWER FLOW IN AIR FOR THE DIPOLE IN AIR

In the preceding chapter we found the approximate integral expressions for the Cartesian components of the Hertzian vector in the air. We also defined certain fundamental integrals which, when evaluated, can be used to find the explicit form of the Hertzian vector and the field components by differentiation.

In this chapter we first evaluate approximately these fundamental integrals, find the components of the Hertzian vector, and finally, the components of the radiation field and the Poynting vector.

7.1 EVALUATION OF THE FUNDAMENTAL INTEGRALS

The fundamental integrals of the previous chapter are very similar to those of Chapter 4, the only essential difference being the absence of the factor exp (ish). The reason for the factor "h" not appearing in the integrals under consideration is the fact in the present problem the corresponding factor exp $(is_o h)$ was combined with exp $(is_o z)$ by means of a proper shift in the coordinate system, and thus, absorbed in the radial distance r or r_o. The basic method that has formulated in Chapter 4 can be used here effectively and the evaluation of the present integrals can be accomplished almost by inspection.

First we consider the first two integrals in equation (6.22a). These

forms are well-known (2,p.22) and the results can be written at once. We have

$$\iint_{-\infty}^{\infty} \frac{e^{i(\alpha_1 x + \alpha_2 y + s_0|z-h|)}}{s_0}\, d\alpha_1 d\alpha_2 = -2\pi i \frac{e^{ik_0 r_0}}{r_0} \tag{7.1a}$$

$$\iint_{-\infty}^{\infty} \frac{e^{i[\alpha_1 x + \alpha_2 y + s_0(z+h)]}}{s_0}\, d\alpha_1 d\alpha_2 = -2\pi i \frac{e^{ik_0 r}}{r}\,. \tag{7.1b}$$

Next, we focus our attention on the integral V_1 in equation (6.27a) and compare it with the integral U_1 in equation (3.44). We note that

$$V_1 = U_1 \ (h = 0). \tag{7.2}$$

Thus, by (4.61)

$$V_1 \sim -\frac{i2}{k_0}\left(\frac{\cos\theta}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}\right)\frac{e^{ik_0 r}}{r}\,. \tag{7.3a}$$

Similarly comparing V_2 in (6.27b) and U_3 in (3.44c) and (4.74), we can write

$$V_2 \sim -\frac{i2}{k_0}\left[\frac{\cos\theta}{(n^2\cos\theta + \sqrt{n^2 - \sin^2\theta})^2}\right]\frac{e^{ik_0 r}}{r}\,. \tag{7.3b}$$

Analogously with V_3 and V_4

$$V_3 \sim -\frac{i2}{k_0}\left[\frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}\,(n^2\cos\theta + \sqrt{n^2 - \sin^2\theta})^2}\right]\frac{e^{ik_0 r}}{r} \tag{7.3c}$$

$$V_4 \sim -\frac{i2}{k_0}\left[\frac{\cos\theta}{(\cos\theta + \sqrt{n^2 - \sin^2\theta})(n^2\cos\theta + \sqrt{n^2 - \sin^2\theta})}\right]\frac{e^{ik_0 r}}{r}\,. \tag{7.3d}$$

Finally, comparing V_5 in equation (6.27e) and U_2 in equation (3.44b), we obtain using (4.70)

$$V_5 \sim -\frac{1}{k_0}\left\{\frac{\cos\theta}{\sin\theta\sin\varphi\sqrt{n^2 - \sin^2\theta}\,(\cos\theta + \sqrt{n^2 - \sin^2\theta})(n^2\cos\theta + \sqrt{n^2 - \sin^2\theta})}\right\}\frac{e^{ik_0 r}}{r} \tag{7.3e}$$

7.2 THE HERTZIAN VECTOR AND THE FIELDS

Having found the leading terms of the asymptotic expansions for V_1 through V_5 we shall now use these results to write down the components of the Hertzian vector in explicit form and later the components of the radiation field.

<u>7.2 a The Hertzian vector</u>—Applying the expressions for the partial derivatives from the results of (4.76), we rewrite (6.22a) and (6.22b) as follows:

$$\Pi_{xo} = \frac{i\,m}{4\pi\omega\mu_o}\left\{\frac{e^{ik_or_o}}{r_o} - \frac{e^{ik_or}}{r} + ik_on^2V_1 + i\eta k_o\sin\theta\left[i\sin\varphi V_2 - \sin\theta\cos\theta\cos^2\varphi\, V_5\right]\right\} \tag{7.4a}$$

$$\Pi_{zo} = \frac{-mk_o\cos\varphi\sin\theta}{4\pi\omega\mu_o}\left\{(1-n^2)V_4 - \eta\left[i\sin\varphi\sin\theta\, V_3 + (\sin^2\varphi\sin^2\theta + \cos^2\theta)\,V_5\right]\right\}. \tag{7.4b}$$

Now incorporating the results of (7.3), we obtain for the radiation field

$$\Pi_{xo} = \frac{i\,m}{4\pi\omega\mu_o}\left\{-\frac{e^{ik_or_o}}{r_o} + \frac{e^{ik_or}}{r} + \frac{2n^2\cos\theta}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}}\cdot\frac{e^{ik_or}}{r} + \frac{i\eta\sin\theta\cos\theta}{(n^2\cos\theta + \sqrt{n^2-\sin^2\theta})}\left[\frac{2\sin\varphi}{n^2\cos\theta+\sqrt{n^2-\sin^2\theta}} + \frac{\cos\theta\cos^2\varphi}{\sin\varphi\sqrt{n^2-\sin^2\theta}(\cos\theta+\sqrt{n^2-\sin^2\theta})}\right]\frac{e^{ik_or}}{r}\right\} \tag{7.5a}$$

and

$$\Pi_{zo} = \frac{i\,m\cos\varphi\sin\theta\cos\theta}{4\pi\omega\mu_o(n^2\cos\theta+\sqrt{n^2-\sin^2\theta})}\left\{2(\cos\theta - \sqrt{n^2-\sin^2\theta}) - \frac{i\eta}{\sqrt{n^2-\sin^2\theta}}\left[\frac{2\sin\varphi\sin\theta}{n^2\cos\theta+\sqrt{n^2-\sin^2\theta}} - \frac{\sin^2\varphi\sin^2\theta+\cos^2\theta}{\sin\varphi\sin\theta(\cos\theta+\sqrt{n^2-\sin^2\theta})}\right]\right\}\frac{e^{ik_or}}{r}. \tag{7.5b}$$

<u>7.2 b The radiation field</u>—We shall now write the explicit forms of the field components. Using the results of (5.4) as well as those of (7.5), we obtain

$$E_{\theta o} = \frac{-imk_o}{2\pi}\left\{ \frac{\sin\varphi}{2}\left(\frac{e^{ik_o r}}{r} - \frac{e^{ik_o r_o}}{r_o}\right) + \frac{n^2\cos\theta\,\sin\varphi}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}}\;\frac{e^{ik_o r}}{r} \right.$$

$$+ \frac{i\eta\sin\theta\cos\theta}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}}\left[\frac{\sin^2\varphi}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}}\right. \tag{7.6a}$$

$$\left.\left. + \frac{\cos\theta\,\cos^2\varphi}{2\sqrt{n^2-\sin^2\theta}\,(\cos\theta + \sqrt{n^2-\sin^2\theta}\,)}\right]\frac{e^{ik_o r}}{r}\right\}$$

and

$$E_{\varphi o} = \frac{-ik_o m\cos\varphi\,\cos\theta}{4\pi}\left\{ -\frac{e^{ik_o r_o}}{r_o} + \frac{e^{ik_o r}}{r}\right.$$

$$+ \frac{2\sqrt{n^2-\sin^2\theta}\,(\sin^2\theta + \cos\theta\sqrt{n^2-\sin^2\theta}\,)}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}}\cdot\frac{e^{ik_o r}}{r}$$

$$+ \frac{i\eta\sin\varphi\,\sin\theta}{\sqrt{n^2-\sin^2\theta}\,(n^2\cos\theta + \sqrt{n^2-\sin^2\theta})}\left[\frac{2(\sin^2\theta + \cos\theta\sqrt{n^2-\sin^2\theta}\,)}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}}\right. \tag{7.6b}$$

$$\left.\left. - \frac{1}{\cos\theta + \sqrt{n^2-\sin^2\theta}}\right]\frac{e^{ik_o r}}{r}\right\}.$$

The magnetic field components are related to the electric field components as in (5.9), that is,

$$\frac{E_{\theta o}}{H_{\varphi o}} = -\frac{E_{\varphi o}}{H_{\theta o}} = Z_o \tag{7.7}$$

where Z_o is the free space impedance.

7.3 THE POWER FLOW

The radiation field components found in the previous section can now be used to obtain the time-averaged Poynting vector given by (5.14). Thus, we obtain

$$S_{\varphi o} = \frac{m^2 k_o^2}{8\pi^2 Z_o} \left\{ \left| G_{\theta o}(\theta,\varphi) \right|^2 + \cos^2\varphi \cos^2\theta \left| G_\varphi(\theta,\varphi) \right|^2 \right\} \tag{7.8}$$

where

$$G_{\theta o} = \frac{\sin\varphi}{2}\left(\frac{e^{ik_o r}}{r} - \frac{e^{ik_o r_o}}{r_o}\right) + \frac{\cos\theta}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}} \left\{ n^2 \sin\varphi \right.$$

$$\left. + i\eta \sin\theta \left[\frac{\sin^2\varphi}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}} + \frac{\cos\theta\cos^2\varphi}{2\sqrt{n^2-\sin^2\theta}(\cos\theta + \sqrt{n^2-\sin^2\theta})} \right] \frac{e^{ik_o r}}{r} \right\} \tag{7.9a}$$

$$G_{\varphi o} = \left(\frac{e^{ik_o r}}{r} - \frac{e^{ik_o r_o}}{r_o}\right) + \frac{1}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}} \left\{ \sqrt{n^2-\sin^2\theta}(\sin^2\theta + \cos\theta\sqrt{n^2-\sin^2\theta}) \right.$$

$$\left. + \frac{i\eta\sin\varphi\sin\theta}{\sqrt{n^2-\sin^2\theta}} \left[\frac{\sin^2\theta + \cos\theta\sqrt{n^2-\sin^2\theta}}{n^2\cos\theta + \sqrt{n^2-\sin^2\theta}} - \frac{1}{2(\cos\theta + \sqrt{n^2-\sin^2\theta})} \right] \right\} \frac{e^{ik_o r}}{r}. \tag{7.9b}$$

Again we note that when $h = 0$ the Poynting vector in this case is identical to that in equation (5.15) and corresponding to the case when the dipole is situated in the magnetoplasma.

7.4 CLOSURE

In this chapter we have completed the approximate high frequency solution to the problem of a horizontal magnetic dipole in air in the presence of a magnetoplasma half-space. In particular, we found all of the radiation field components in the air and were able to separate definitely the contributions of the plasma's anisotropy.

As a partial check on the results of this chapter, as well as those of

Chapter 5, we noted that the corresponding results were identical when the depth of the source's burial in each case was allowed to approach zero, i.e., when the source was situated at the interface.

Although no numerical example was considered in this problem, the results can be expected to be substantially the same as those of Chapter 5 and therefore, the conclusions drawn in that chapter also apply here.

PART II

FIELD OF ELECTRIC CURRENT LINE SOURCES IN MAGNETOPLASMA WITH A SEPARATION BOUNDARY

CHAPTER 8

RIGOROUS FORMULATION OF THE PROBLEM OF AN ELECTRIC CURRENT LINE SOURCE WHEN THE STEADY MAGNETIC FIELD IS PERPENDICULAR TO IT

In this chapter we shall be concerned with the finding of appropriate integral representations for the Cartesian components of the field vectors for the magnetoplasma- and air-half- spaces. This will entail review of the fundamental field equations and the definition of the source of electromagnetic waves.

8.1 STATEMENT OF THE PROBLEM

The geometry of the problem is shown in Fig. 8.1. The horizontal plane $z = 0$ coincides with the interface between the anisotropic homogenous plasma and air. Again for convenience, we shall call the plasma medium (1) and the air medium (0). As before, we assume that both media have the same magnetic inductive capacity of free space, μ_0. The steady magnetic field H_{DC} will, in this case, be oriented perpendicular to the line source whereas the plane formed by the line source and the direction of the steady magnetic field is parallel to the interface.

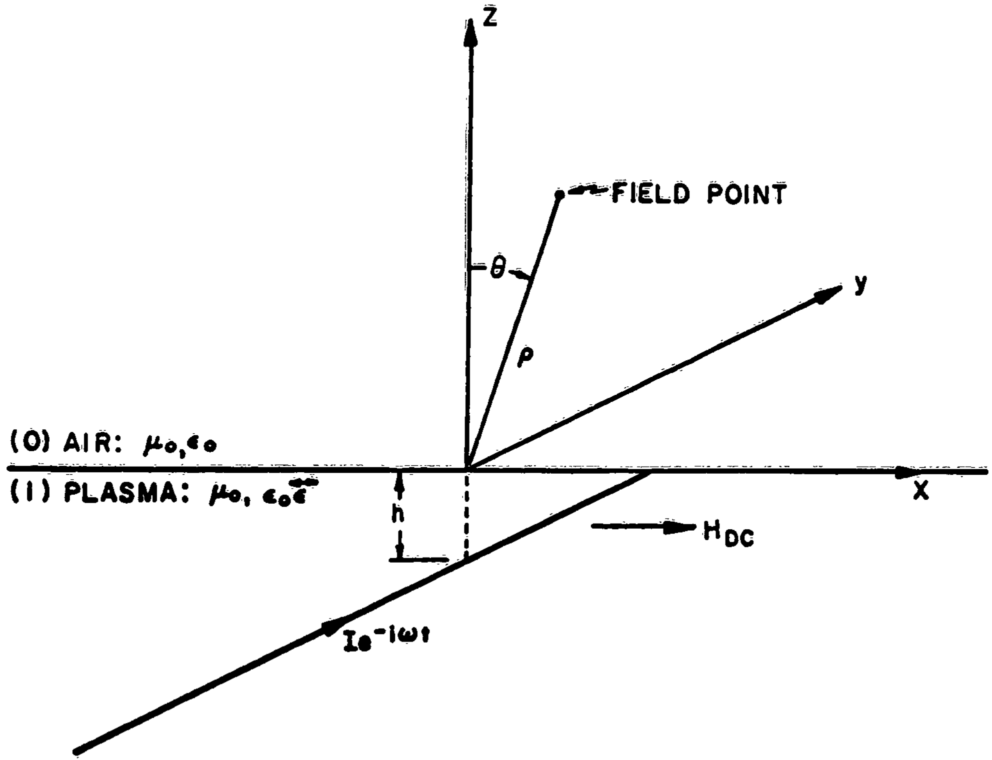

Figure 8.1 - Geometry of the Problem of an Electric Current Line Source When the Steady Magnetic Field is Perpendicular

8.2 FUNDAMENTAL EQUATIONS

The definition of the present boundary value problem implies solution to Maxwell's equations subject to the usual boundary conditions at the interface and proper behavior at infinity. The use of the auxiliary vector potentials does not seem to simplify the problem any. Thus, we shall operate with the field components directly.

8.2 a The nature of the source--For the purpose of this problem it will be assumed that the source of the electromagnetic waves consists of a very thin straight wire of infinite extent carrying an alternating current $Ie^{-i\omega t}$. To localize the source properly we write for the electric current density

$$\vec{J} = I\,\delta(x)\,\delta(z+h)\,\vec{1}_y \qquad (8.1)$$

where $\delta()$ is the Dirac delta function (7,p.43).

8.2 b The field equations--We assumed previously that our source of electromagnetic waves in this problem may be regarded as an electric current line singularity at a point $x = 0$ and $z = -h$. Then for the magnetoplasma region the appropriate form of Maxwell's equation is

$$\begin{aligned} \vec{\nabla} \times \vec{E}_1 &= i\omega\mu_0\vec{H}_1 \\ \vec{\nabla} \times \vec{H}_1 &= -i\omega\epsilon_0\tilde{\epsilon}\,\vec{E}_1 + I\,\delta(x)\,\delta(z+h)\,\vec{1}_y \\ \vec{\nabla}\cdot\vec{D}_1 &= \rho \\ \vec{\nabla}\cdot\vec{B}_1 &= 0 \end{aligned} \qquad (8.2)$$

where ρ is the electric charge density related to the electric current by the continuity equation

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}. \qquad (8.3)$$

To obtain the wave equation we perform a curl operation on the first equation of (8.2) and then substitute for $\vec{\nabla} \times \vec{H}$ from the second and obtain

$$- \vec{\nabla} \times \vec{\nabla} \times \vec{E}_1 + k_o^2 \overleftrightarrow{\epsilon} \vec{E}_1 = - i\omega\mu_o I \delta(x) \delta(z + h) \vec{1}_y \, . \qquad (8.4)$$

Using the components of the tensor as in equation (2.8) and carrying out the necessary algebra noting that $\partial_y = 0$, one obtains a system of simultaneous differential equations

$$\begin{bmatrix} \zeta k_o^2 + \partial_z^2 & 0 & -\partial_x \partial_z \\ 0 & \epsilon k_o^2 + \partial_x^2 + \partial_z^2 & i\eta k_o^2 \\ -\partial_x \partial_z & -i\eta k_o^2 & \epsilon k_o^2 + \partial_x^2 \end{bmatrix} \begin{bmatrix} E_{x1} \\ E_{y1} \\ E_{z1} \end{bmatrix} = -i\omega\mu_o I \delta(x) \delta(z+h) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad (8.5)$$

which we shall leave in this form for the time being. From the first equation of (8.2) we note that the magnetic field is expressible in terms of the electric field as follows:

$$\begin{bmatrix} H_{x1} \\ H_{y1} \\ H_{z1} \end{bmatrix} = \frac{1}{i\omega\mu_o} \begin{bmatrix} 0 & -\partial_z & 0 \\ \partial_z & 0 & -\partial_x \\ 0 & \partial_x & 0 \end{bmatrix} \begin{bmatrix} E_{x1} \\ E_{y1} \\ E_{z1} \end{bmatrix} \qquad (8.6)$$

In the air-region the appropriate form of the Maxwell's equation is

$$\begin{aligned} \vec{\nabla} \times \vec{E}_o &= i\omega\mu_o \vec{H}_o \\ \vec{\nabla} \times \vec{H}_o &= -i\omega\epsilon_o \vec{E}_o \\ \vec{\nabla} \cdot \vec{D}_o &= 0 \\ \vec{\nabla} \cdot \vec{B}_o &= 0 \end{aligned} \qquad (8.7)$$

and the Cartesian components of the electric and magnetic fields satisfy the vector wave equation

$$\left\{ \nabla^2 + k_o^2 \right\} \cdot \left\{ \begin{matrix} \vec{E}_o \\ \vec{H}_o \end{matrix} \right\} = 0 \, . \qquad (8.8)$$

8.3 FOURIER INTEGRAL REPRESENTATION IN CARTESIAN COORDINATES

The formulation of the present boundary value problem can be simplified a great deal by expressing the field components in magnetoplasma and in the air in terms of their double Fourier integral representation in Cartesian coordinates in the transform space as well as in the configuration space.

Thus, we introduce a double Fourier transform pair defined by

$$\tilde{\tilde{F}}(\alpha_1, \alpha_3) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} F(x,z)\, e^{-i(\alpha_1 x + \alpha_3 z)}\, dx\, dz \tag{8.9}$$

and

$$F(x,z) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \tilde{\tilde{F}}(\alpha_1, \alpha_3)\, e^{i(\alpha_1 x + \alpha_3 z)}\, d\alpha_1\, d\alpha_3 \,. \tag{8.10}$$

In what follows we shall also need the transforms of the derivatives. These can be obtained by integrating by parts. As we showed in Chapter 2 for the triple transform, the vanishing of the integrated part at the upper and lower limits is assured providing the radiation condition is satisfied. Thus, assuming that the radiation condition is satisfied we can establish the following correspondences:

$$\begin{aligned} \partial_x F &\Longleftrightarrow i\alpha_1 \tilde{\tilde{F}} \\ \partial_z F &\Longleftrightarrow i\alpha_3 \tilde{\tilde{F}} \,. \end{aligned} \tag{8.11}$$

<u>8.3 a The particular integral corresponding to the source</u>—To transform the inhomogeneous system of simultaneous differential equations (8.5), one multiplies both sides by $\exp\{i(\alpha_1 x + \alpha_3 z)\}$ and integrates with respect to the real variables x and z between $-\infty$ and $+\infty$. The right-hand side of (8.5) yields at once

$$\iint_{-\infty}^{\infty} \delta(x)\,\delta(z+h)\, e^{-i(\alpha_1 x + \alpha_3 z)}\, dx\, dz = e^{i\alpha_3 h} \tag{8.12}$$

and the left-hand side transforms according to equations (8.9) and (8.11). Thus, one obtains

$$\begin{bmatrix} \zeta k_0^2 - \alpha_3^2 & 0 & \alpha_1\alpha_3 \\ 0 & \epsilon k_0^2 - \alpha_1^2 - \alpha_3^2 & i\eta k_0^2 \\ \alpha_1\alpha_3 & -i\eta k_0^2 & \epsilon k_0^2 - \alpha_1^2 \end{bmatrix} \begin{bmatrix} \tilde{\tilde{E}}_{x_1} \\ \tilde{\tilde{E}}_{y_1} \\ \tilde{\tilde{E}}_{z_1} \end{bmatrix} = \frac{-i\omega\mu_0 I}{2\pi} \begin{bmatrix} 0 \\ e^{i\alpha_3 h} \\ 0 \end{bmatrix} \tag{8.13}$$

which can be immediately reduced to

$$\begin{bmatrix} \zeta k_0^2 - \alpha_3^2 & 0 & \alpha_1\alpha_3 \\ 0 & (\alpha_3^2 - s_1^2)(\alpha_3^2 - s_2^2) & 0 \\ 0 & -i\kappa(\zeta k_0^2 - \alpha_3^2) & \frac{\zeta}{\epsilon}(\epsilon k_0^2 - \alpha_1^2) - \alpha_3^2 \end{bmatrix} \begin{bmatrix} \tilde{\tilde{E}}_{x_1} \\ \tilde{\tilde{E}}_{y_1} \\ \tilde{\tilde{E}}_{z_1} \end{bmatrix} = \frac{-i\omega\mu_0 I}{2\pi}\left[\frac{\zeta}{\epsilon}(\epsilon k_0^2 - \alpha_1^2) - \alpha_3^2\right] \begin{bmatrix} 0 \\ e^{i\alpha_3 h} \\ 0 \end{bmatrix} \tag{8.14}$$

where now

$$s_{1,2}^2 = \frac{1}{2}\left\{(\chi+\zeta)k_0^2 - \frac{\epsilon+\zeta}{\epsilon}\alpha_1^2 \pm \sqrt{\left[(\chi-\zeta)k_0^2 - \frac{\epsilon-\zeta}{\epsilon}\alpha_1^2\right]^2 + 4\kappa^2\zeta k_0^2\alpha_1^2}\right\} \tag{8.15}$$

We compare (8.15) with (2.41) and observe that the former is the same as the latter with $\alpha_2 = 0$. From (8.7) we can write each transformed component of the electric field as follows:

$$\tilde{\tilde{E}}_{y_1}^{(P)} = -\frac{i\omega\mu_0 I}{2\pi\epsilon}\left[\frac{\zeta(\epsilon k_0^2 - \alpha_1^2) - \epsilon\alpha_3^2}{(\alpha_3^2 - s_1^2)(\alpha_3^2 - s_2^2)}\right] e^{i\alpha_3 h} \tag{8.16a}$$

$$\tilde{\tilde{E}}_{z_1}^{(P)} = -\frac{i\omega\mu_0 I}{2\pi\epsilon}\left[\frac{i\eta(\zeta k_0^2 - \alpha_3^2)}{(\alpha_3^2 - s_1^2)(\alpha_3^2 - s_2^2)}\right] e^{i\alpha_3 h} \tag{8.16b}$$

$$\tilde{\tilde{E}}_{x_1}^{(P)} = \frac{i\omega\mu_0 I}{2\pi\epsilon}\left[\frac{i\eta\alpha_1\alpha_3}{(\alpha_3^2 - s_1^2)(\alpha_3^2 - s_2^2)}\right] e^{i\alpha_3 h}. \tag{8.16c}$$

The inversion with respect to the α_3 transform variable can be performed immediately. According to (2.53) and (2.54) we obtain

$$\tilde{E}_{y_1}^{(P)} = \frac{\omega\mu_0 I}{2\sqrt{2\pi}\epsilon}\left\{\frac{\zeta(\epsilon k_0^2 - \alpha_1^2) - \epsilon s_1^2}{s_1(s_1^2 - s_2^2)} e^{is_1|z+h|} - \frac{\zeta(\epsilon k_0^2 - \alpha_1^2) - \epsilon s_2^2}{s_2(s_1^2 - s_2^2)} e^{is_2|z+h|}\right\} \tag{8.17a}$$

$$\tilde{E}^{(p)}_{z1} = \frac{i\kappa\omega\mu_0 I}{2\sqrt{2\pi}} \left\{ \frac{\zeta k_0^2 - s_1^2}{s_1(s_1^2 - s_2^2)} e^{is_1|z+h|} - \frac{\zeta k_0^2 - s_2^2}{s_2(s_1^2 - s_2^2)} e^{is_2|z+h|} \right\} \tag{8.17b}$$

$$\tilde{E}^{(p)}_{x1} \underset{z+h\gtrless 0}{=} \mp \frac{i\kappa\omega\mu_0 I}{2\sqrt{2\pi}} \alpha_1 \left\{ \frac{e^{is_1|z+h|}}{s_1^2 - s_2^2} - \frac{e^{is_2|z+h|}}{s_1^2 - s_2^2} \right\} \tag{8.17c}$$

Finally, we invert with respect to the α_1 transform variable and obtain the desired representations

$$E^{(p)}_{y1} = \frac{\omega\mu_0 I}{4\pi\epsilon} \int_{-\infty}^{\infty} \left\{ \frac{\zeta(\epsilon k_0^2 - \alpha_1^2) - \epsilon s_1^2}{s_1(s_1^2 - s_2^2)} e^{is_1|z+h|} - \frac{\zeta(\epsilon k_0^2 - \alpha_1^2) - \epsilon s_2^2}{s_2(s_1^2 - s_2^2)} e^{is_2|z+h|} \right\} e^{i\alpha_1 x} d\alpha_1 \tag{8.18a}$$

$$E^{(p)}_{z1} = \frac{i\kappa\omega\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \left\{ \frac{\zeta k_0^2 - s_1^2}{s_1(s_1^2 - s_2^2)} e^{is_1|z+h|} - \frac{\zeta k_0^2 - s_2^2}{s_2(s_1^2 - s_2^2)} e^{is_2|z+h|} \right\} e^{i\alpha_1 x} d\alpha_1 \tag{8.18b}$$

$$E^{(p)}_{x1} \underset{z+h\gtrless 0}{=} \mp \frac{\kappa\omega\mu_0 I}{4\pi} \partial_x \int_{-\infty}^{\infty} \left\{ \frac{e^{is_1|z+h|} - e^{is_2|z+h|}}{s_1^2 - s_2^2} \right\} e^{i\alpha_1 x} d\alpha_1 \tag{8.18c}$$

<u>8.3 b The complementary field in the plasma</u>--In the preceding section we obtained the "particular integrals" of the system of equations (8.13) which represent primary excitation due to the source. To satisfy the boundary conditions of the problem, we shall need an appropriate complementary solution of the homogeneous system of (8.13).

From (8.14) it is clear that $\vec{\tilde{E}}^{(c)}_1$ satisfies

$$(\alpha_3^2 - s_1^2)(\alpha_3^2 - s_2^2)\,\vec{\tilde{E}}^{(c)}_1 = 0 \tag{8.19}$$

where in the above equation α_3 is looked upon as a differential operator with respect to the z-coordinate. The solutions to (8.19) can be written immediately

$$\begin{bmatrix} \tilde{E}^{(c)}_{x1} \\ \tilde{E}^{(c)}_{y1} \\ \tilde{E}^{(c)}_{z1} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} \begin{bmatrix} e^{-is_1 z} \\ e^{-is_2 z} \end{bmatrix} \tag{8.20}$$

where we discarded solutions with positive exponentials since we can have only waves going away from the interface upon reflection. The coefficients C_{ij} can be found by a method identical to the one of Section 2.3. We obtain

$$C_{11} = \frac{i\eta\alpha_1 s_1}{\zeta(\epsilon k_o^2 - \alpha_1^2) - \epsilon s_1^2} C_{21} \tag{8.21a}$$

$$C_{31} = \frac{i\eta(\zeta k_o^2 - s_1^2)}{\zeta(\epsilon k_o^2 - \alpha_1^2) - \epsilon s_1^2} C_{21} \tag{8.21b}$$

and

$$C_{12} = \frac{i\eta\alpha_1 s_2}{\zeta(\epsilon k_o^2 - \alpha_1^2) - \epsilon s_2^2} C_{22} \tag{8.21c}$$

$$C_{32} = \frac{i\eta(\zeta k_o^2 - s_2^2)}{\zeta(\epsilon k_o^2 - \alpha_1^2) - \epsilon s_2^2} C_{22} \ . \tag{8.21d}$$

For convenience in what follows we shall normalize the coefficients C_{ij} and put

$$C_{21} = \frac{\omega\mu_o I}{2\sqrt{2\pi}\,\epsilon} A_1 \tag{8.22a}$$

$$C_{22} = \frac{\omega\mu_o I}{2\sqrt{2\pi}\,\epsilon} A_2 \ . \tag{8.22b}$$

Now we invert with respect to the α_1 transform variable and obtain the desired representations

$$E_{y_1}^{(c)} = \frac{\omega\mu_o I}{4\pi\epsilon}\int_{-\infty}^{\infty}\left\{A_1 e^{-is_1 z} + A_2 e^{-is_2 z}\right\} e^{i\alpha_1 x} d\alpha_1 \tag{8.23a}$$

$$E_{z_1}^{(c)} = \frac{i\kappa\omega\mu_o I}{4\pi}\int_{-\infty}^{\infty}\left\{\frac{(\zeta k_o^2 - s_1^2)A_1 e^{-is_1 z}}{\zeta(\epsilon k_o^2 - \alpha_1^2) - \epsilon s_1^2} + \frac{(\zeta k_o^2 - s_2^2)A_2 e^{-is_2 z}}{\zeta(\epsilon k_o^2 - \alpha_1^2) - \epsilon s_2^2}\right\} e^{i\alpha_1 x} d\alpha_1 \tag{8.23b}$$

$$E_{x_1}^{(c)} = \frac{\kappa\omega\mu_o I}{4\pi}\partial_x\int_{-\infty}^{\infty}\left\{\frac{s_1 A_1 e^{-is_1 z}}{\zeta(\epsilon k_o^2 - \alpha_1^2) - \epsilon s_1^2} + \frac{s_2 A_2 e^{-is_2 z}}{\zeta(\epsilon k_o^2 - \alpha_1^2) - \epsilon s_2^2}\right\} e^{i\alpha_1 x} d\alpha_1 \tag{8.23c}$$

8.3 c The field in the air—For the air region we choose the following solution to the wave equation (8.8):

$$E_{yo} = \frac{\omega\mu_o I}{4\pi\epsilon}\int_{-\infty}^{\infty} B_1\, e^{\,i(\alpha_1 x + s_o z)}\, d\alpha_1 \tag{8.24a}$$

$$E_{zo} = \frac{i\kappa\omega\mu_o I}{4\pi}\, \partial_x^2 \int_{-\infty}^{\infty} \frac{B_2}{s_o}\, e^{\,i(\alpha_1 x + s_o z)}\, d\alpha_1 \tag{8.24b}$$

$$E_{xo} = \frac{\kappa\omega\mu_o I}{4\pi}\, \partial_x \int_{-\infty}^{\infty} B_2\, e^{\,i(\alpha_1 x + s_o z)}\, d\alpha_1 \,. \tag{8.24c}$$

It can be readily shown that the above field components satisfy the divergence equation $\vec{\nabla}\cdot\vec{E}_o = 0$.

8.4 THE BOUNDARY CONDITIONS

In the preceding sections we found field components in the plasma and air-regions that are solutions to Maxwell's equations. Moreover, in solving Maxwell's equations, we chose such solutions for field representation, that have proper behavior at infinity by requiring that the imaginary part of the pertinent exponents be non-negative. In addition, the field components contain certain, thus far, undetermined coefficients which upon imposition of the boundary conditions will be determined and thus render the solution unique.

8.4 a Statement of the boundary conditions—The boundary conditions to be satisfied by the Cartesian components of the field vectors require continuity of the tangential components of the electric and magnetic fields at the interface $z = 0$. This implies the following:

$$H_{x0} = H_{x1}$$
$$H_{y0} = H_{y1} \qquad (8.25)$$
$$E_{x0} = E_{x1}$$
$$E_{y0} = E_{y1} \; .$$

We rewrite the above equations in terms of the electric field only, getting

$$E_{x0} = E_{x1} \qquad (8.26a)$$

$$E_{y0} = E_{y1} \qquad (8.26b)$$

$$\partial_z E_{y0} = \partial_z E_{y1} \qquad (8.26c)$$

$$\partial_z E_{x0} - \partial_x E_{z0} = \partial_z E_{x1} - \partial_x E_{z1} \; . \qquad (8.26d)$$

These equations can be simplified to a more convenient form

$$0 = i\,\partial_z \check{E}_{y1} + s_0 \check{E}_{y1} \qquad (8.27a)$$

$$0 = i\,\partial_z \check{E}_{x1} + \frac{s_0^2 + \alpha_1^2}{s_0} \check{E}_{x1} + \alpha_1 \check{E}_{z1} \qquad (8.27b)$$

where $\vee$ denotes the field representations with the integral sign removed.

<u>8.4 b Application of the boundary condition</u>—Performing the necessary algebraic operation to satisfy equations (8.27a) and (8.27b) gives a set of simultaneous algebraic equations in two unknowns as follows:

$$\begin{bmatrix} s_1 + s_0 & s_2 + s_0 \\ \frac{s_1 + \zeta s_0}{P_E(s_1)} & \frac{s_2 + \zeta s_0}{P_E(s_2)} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \frac{(s_1 - s_0) P_E(s_1)}{s_1 (s_1^2 - s_2^2)} e^{i s_1 h} - \frac{(s_2 - s_0) P_E(s_2)}{s_2 (s_1^2 - s_2^2)} e^{i s_2 h} \\ \frac{s_1 - \zeta s_0}{s_1 (s_1^2 - s_2^2)} e^{i s_1 h} - \frac{s_2 - \zeta s_0}{s_2 (s_1^2 - s_2^2)} e^{i s_2 h} \end{bmatrix} \qquad (8.28)$$

where

$$P_E(s_{1,2}) = \zeta(\epsilon k_0^2 - \alpha_1^2) - \epsilon s_{1,2}^2 \; . \qquad (8.29)$$

We solve the above system of equations using Cramer's rule. We find

$$A_1 = \frac{P_E(s_1)\left[(s_1-s_0)(s_2+\zeta s_0)P_E(s_1)-(s_2+s_0)(s_1-\zeta s_0)P_E(s_2)\right]}{s_1(s_1^2-s_2^2)N_E}\,e^{is_1h}$$
$$-\frac{P_E(s_1)P_E(s_2)\left[(s_2-s_0)(s_2+\zeta s_0)-(s_2+s_0)(s_2-\zeta s_0)\right]}{s_2(s_1^2-s_2^2)N_E}\,e^{is_2h} \tag{8.30a}$$

and

$$A_2 = \frac{P_E(s_2)\left[(s_2-s_0)(s_1+\zeta s_0)P_E(s_2)-(s_1+s_0)(s_2-\zeta s_0)P_E(s_1)\right]}{s_2(s_1^2-s_2^2)N_E}\,e^{is_2h} \tag{8.30b}$$
$$-\frac{P_E(s_1)P_E(s_2)\left[(s_1-s_0)(s_1+\zeta s_0)-(s_1+s_0)(s_1-\zeta s_0)\right]}{s_1(s_1^2-s_2^2)N_E}\,e^{is_1h}$$

where

$$N_E = (s_1+s_0)(s_2+\zeta s_0)\,P_E(s_1) - (s_2+s_0)(s_1+\zeta s_0)\,P_E(s_2)\,. \tag{8.31}$$

To find B_1 and B_2 we use (8.26a) and (8.26b) and the above results and obtain

$$B_1 = -2\epsilon\left\{\frac{P_E(s_1)(s_2+\zeta s_0)e^{is_1h} - P_E(s_2)(s_1+\zeta s_0)e^{is_2h}}{N_E}\right\} \tag{8.32a}$$

and

$$B_2 = 2\zeta s_0\left\{\frac{(s_2+s_0)e^{is_1h}-(s_1+s_0)e^{is_2h}}{N_E}\right\}. \tag{8.32b}$$

8.5 FIELD COMPONENTS IN THE AIR IN CYLINDRICAL COORDINATES

Having found rigorous expressions for the field components in the Cartesian coordinates in both of the regions, we can now transform to cylindrical coordinates in which the field components will have somewhat simpler form.

We transform to cylindrical coordinates in both configuration and transform spaces using

$$\begin{aligned}\alpha_1 &= k_0\sin\beta\\ z &= \rho\cos\theta\\ x &= \rho\sin\theta\end{aligned} \tag{8.33}$$

As a consequence of the above, we observe that

$$\alpha_1 x + s_o z = k_o \rho \cos(\beta - \theta) \tag{8.34}$$

furthermore,

$$\partial_x = \sin\theta\, \partial_\rho + \frac{\cos\theta}{\rho} \partial_\theta \tag{8.35}$$

$$\partial_z = \cos\theta\, \partial_\rho - \frac{\sin\theta}{\rho} \partial_\theta$$

and

$$\begin{aligned} (\ )_\theta &= -(\ )_z \sin\theta + (\ )_x \cos\theta \\ (\ )_\rho &= \ \ (\ )_z \cos\theta + (\ )_x \sin\theta \end{aligned} \tag{8.36}$$

Using the above relationships, the electric field components can be written

$$E_{yo} = -\frac{\omega\mu_o k_o I}{2\pi} \int_{\Gamma_1} \frac{P_E(s_1)(s_2+\xi s_o)e^{is_1h} - P_E(s_2)(s_1+\xi s_o)e^{is_2h}}{N_E} \cos\beta\, e^{ik_o\rho\cos(\beta-\theta)} d\beta \tag{8.37a}$$

$$E_{\theta o} = \frac{\eta\zeta\omega\mu_o k_o^2 I}{2\pi} \partial_x \int_{\Gamma_1} \frac{(s_2+s_o)e^{is_1h} - (s_1+s_o)e^{is_2h}}{N_E} \cos\beta \cos(\beta-\theta) e^{ik_o\rho\cos(\beta-\theta)} d\beta \tag{8.37b}$$

$$E_{\rho o} = -\frac{\eta\zeta\omega\mu_o k_o^2 I}{2\pi} \partial_x \int_{\Gamma_1} \frac{(s_2+s_o)e^{is_1h} - (s_1+s_o)e^{is_2h}}{N_E} \cos\beta \sin(\beta-\theta) e^{ik_o\rho\cos(\beta-\theta)} d\beta \tag{8.37c}$$

where Γ_1 is the integration path in the complex β-plane as shown in Fig. 4.1. In the above equation it is understood that s_1, s_2, and s_o are all functions of β according to the first equation of (8.33).

8.6 CLOSURE

In the foregoing chapter we formulated rigorously the problem of an electric current line source in a magnetoplasma with a separation boundary. In particular, we found the integral representation for the field components in the Cartesian and cylindrical coordinates in the air-region and in the Cartesian coordinates in the plasma region. Unlike in an ordinary isotropic case, all of the electric and magnetic fields are present in both regions.

CHAPTER 9

RESULTS FOR THE AIR-REGION WHEN THE STEADY MAGNETIC FIELD IS NORMAL TO THE LINE SOURCE

Having obtained the rigorous formal solution to the problem of an electric current line source in a magnetoplasma with a separation boundary in the form of certain definite integrals, we shall endeavor in this chapter to reduce these integrals to a form that is suitable for numerical calculations.

As it was remarked in Chapter 4, these definite integrals do not lend themselves to rigorous evaluation and we shall, therefore, use the method of steepest descents to extract the desired information from them. The method to be used here will be somewhat simpler than in Chapter 4 since only a single integration is necessary.

9.1 SINGULARITIES IN THE β-PLANE

For reasons mentioned in Section 4.1, it is mandatory to know the singularities of the integrands before their approximate evaluation. To this end we shall now examine the singularities of the integrands in (8.36a), (8.36b), and (8.36c).

These integrals when transformed to the β-plane take on the form

$$E_{yo} = \frac{-\omega\mu_o I}{2\pi}\int_{P_1}\frac{R_E(\sigma_1)(\sigma_2+\zeta\cos\beta)e^{ik_oh\sigma_1}-R_E(\sigma_2)(\sigma_1+\zeta\cos\beta)e^{ik_oh\sigma_2}}{\mathcal{H}_E}\cos\beta \cdot e^{ik_o\rho\cos(\beta-\theta)}\,d\beta \tag{9.1a}$$

$$E_{\theta o} = \frac{\kappa\zeta\omega\mu_o I}{2\pi k_o}\partial_x \int_{\Gamma_1} \frac{(\sigma_2+\cos\beta)e^{ik_oh\sigma_1}-(\sigma_1+\cos\beta)e^{ik_oh\sigma_2}}{\mathcal{N}_E}\cos\beta\cos(\beta-\theta) \cdot e^{ik_o\rho\cos(\beta-\theta)}d\beta \tag{9.1b}$$

$$E_{\varphi o} = -\frac{\kappa\zeta\omega\mu_o I}{2\pi k_o}\partial_x \int_{\Gamma_1} \frac{(\sigma_2+\cos\beta)e^{ik_oh\sigma_1}-(\sigma_1+\cos\beta)e^{ik_oh\sigma_2}}{\mathcal{N}_E}\cos\beta\sin(\beta-\theta) \cdot e^{ik_o\rho\cos(\beta-\theta)}d\beta \tag{9.1c}$$

where

$$\sigma_{1,2}^2 = \frac{1}{2}\left\{(\chi+\zeta)-\left(1+\frac{\zeta}{\epsilon}\right)\sin^2\beta \pm \sqrt{\left[(\chi-\zeta)-\left(1-\frac{\zeta}{\epsilon}\right)\sin^2\beta\right]^2+4\kappa^2\zeta\sin^2\beta}\right\} \tag{9.2a}$$

$$\mathcal{N}_E = (\sigma_1+\cos\beta)(\sigma_2+\zeta\cos\beta)P_E(\sigma_1)-(\sigma_2+\cos\beta)(\sigma_1+\zeta\cos\beta)P_E(\sigma_2) \tag{9.2b}$$

$$P_E(\sigma_{1,2}) = \zeta(\epsilon-\sin^2\beta)-\epsilon\sigma_{1,2}^2 \; . \tag{9.2c}$$

We also note that $P_E(\sigma_1)$ and $P_E(\sigma_2)$ can be written in the form

$$P_E(\sigma_{1,2}) = -\left[\frac{\epsilon^2-\eta^2-\epsilon\zeta-(\epsilon-\zeta)\sin^2\beta}{2}\right] \mp \sqrt{\left[\frac{\epsilon^2-\eta^2-\epsilon\zeta-(\epsilon-\zeta)\sin^2\beta}{2}\right]^2+\zeta\eta^2\sin^2\beta} \tag{9.3}$$

and that σ_1 and σ_2 can then be obtained from (9.2c).

<u>9.1 a The location of the poles</u>—The poles of the integrands in (9.1a), (9.1b), and (9.1c) are the zeroes of the denominator $\mathcal{N}_E$. Thus, to obtain the location of the poles we must solve

$$(\sigma_1+\cos\beta)(\sigma_2+\zeta\cos\beta)P_E(\sigma_1)=(\sigma_2+\cos\beta)(\sigma_1+\zeta\cos\beta)P_E(\sigma_2) \tag{9.4}$$

Unfortunately the analytical solution to the above equation is very difficult to find, and one must resort to graphical means to accomplish it.

<u>9.1 b The branch points</u>—The integrands under consideration contain the radicals associated with σ_1 and σ_2 as a result of which the points $\beta = \theta_{B1}$ and $\beta = \theta_{B2}$ at which σ_1 and σ_2 respectively vanish will be branch points. At each point β these integrands can take on four different values depending on which sign we choose for the radicals. It will be convenient here to talk about four sheets of the β-plane (formed by a four-sheeted Riemann surface) on which each integrand is single-valued. These four sheets will be defined as follows:

	Sheet I	Sheet II	Sheet III	Sheet IV
$\mathrm{Im}\{\sigma_1\}$	+	-	+	-
$\mathrm{Im}\{\sigma_2\}$	+	+	-	-

The convergence of the integrals is assured if the path of integration at least begins and ends on Sheet I. For convenience we shall introduce the following "cuts" in the complex β-plane:

$$\mathrm{Im}\{\sigma_{1,2}\} = 0 \; . \tag{9.5}$$

It is clear that the equations for the "cuts" must satisfy

$$\sigma_{1,2}^2 = x^2 \;\; ; \; x^2 \geqslant 0 \; . \tag{9.6}$$

Now the branch points are located at

$$\theta_{B1} = \pm \text{ arc sin } (\sqrt{\epsilon + \eta}) \tag{9.7a}$$

$$\theta_{B2} = \pm \text{ arc sin } (\sqrt{\epsilon - \eta}) \; . \tag{9.7b}$$

We recall that $\sqrt{\epsilon + \eta}$ and $\sqrt{\epsilon - \eta}$ are respectively the indices of refraction of a right- and left-hand circularly polarized plane wave travelling in the direction of the steady magnetic field in the plasma. Furthermore, in terms of a lossless plasma, parameters $(\epsilon + \eta)$ and $(\epsilon - \eta)$ are given by

$$\epsilon \pm \eta = 1 - \frac{p^2}{1 \mp \sigma} \, . \tag{9.8}$$

We observe from the above, that the location of the branch points is strongly dependent on the value of plasma and cyclotron frequencies in the plasma. When $p^2 < 1 \pm \sigma$ both branch points will be found on the real axis and when $p^2 > 1 \pm \sigma$ they will be located on the imaginary axes in the complex β-plane. Finally, when $(1-\sigma) < p < (1+\sigma)$ one branch point will be located on the real axis, whereas, the other on the imaginary axis. Figure 9.1 depicts one of the above discussed situations together with the branch cuts $\mathrm{Im}\{\sigma_1\} = 0$ and $\mathrm{Im}\{\sigma_2\} = 0$ shown dotted and denoted by γ_1 and γ_2 respectively.

With the branch cuts defined we can draw the four-sheeted Riemann surface as shown in Fig. 9.2. We observe that the cut γ_1 joins the Sheets I and II and the cut γ_2 joins the Sheets II and IV.

9.2 FORMULATION OF THE CONTRIBUTIONS TO THE FIELD INTEGRALS

In the previous section we have determined the singularities of the field integrals and, in particular, we found four branch points. In this section we shall determine the various contributions to the field integrals as affected by these singularities.

All of the field integrals are of the form

$$W = \int_{\Gamma_1} F(\beta)\, e^{ik_0\rho\cos(\beta-\theta)}\, d\beta \tag{9.9}$$

and we wish to evaluate these integrals approximately when the distance ρ is large. The saddle point of the integrands occurs when the derivative of the exponent vanishes, i.e., when $\beta = \theta$. By the argument identical to that in Section 4.2, the original path of integration Γ_1 can be deformed to the path of the steepest descents Γ and the integration can be performed along that path.

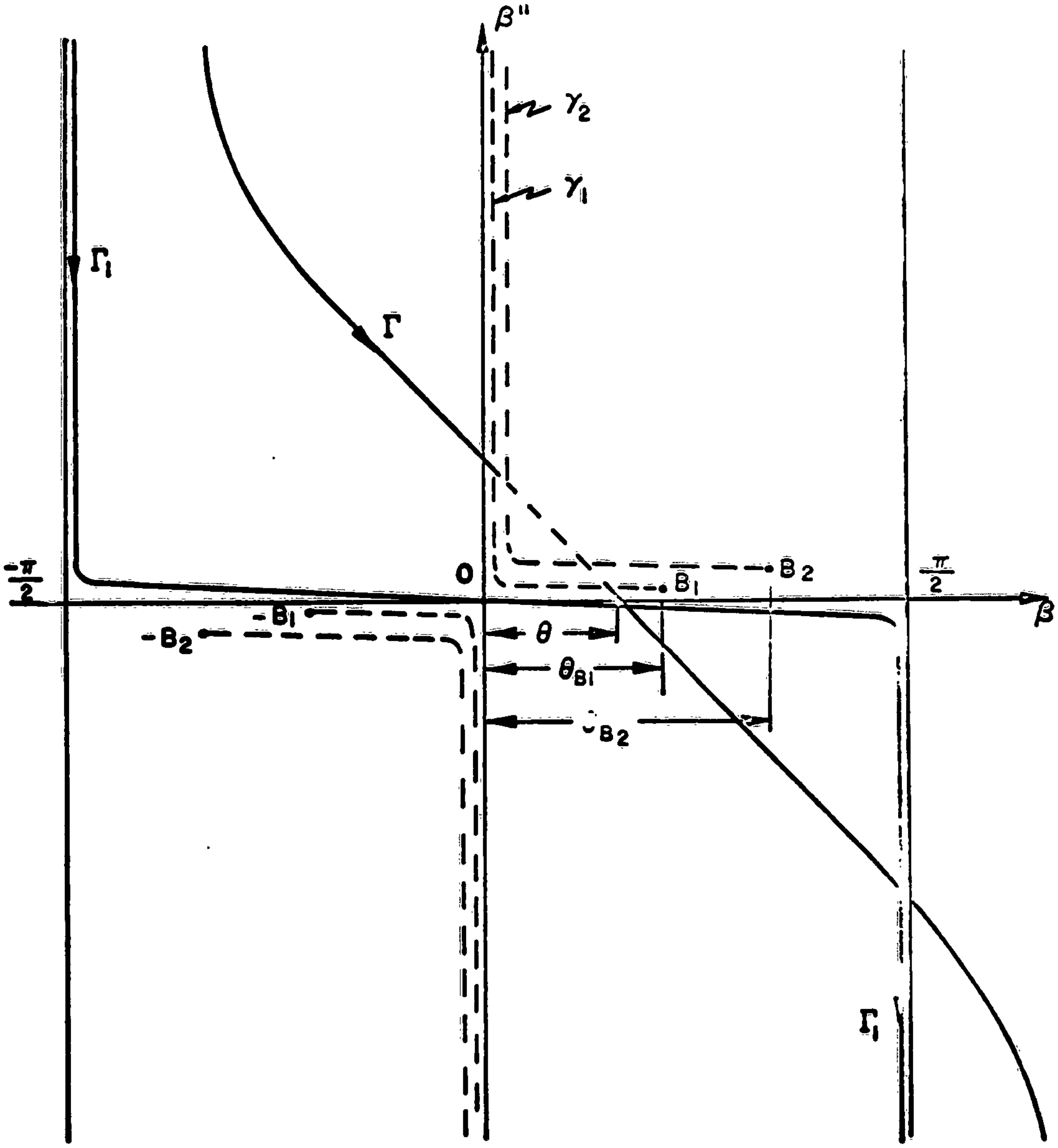

Fig. 9.1 - The Complex β-Plane with the Branch Points B_1 and B_2 When $p^2 < 1 \pm \sigma$

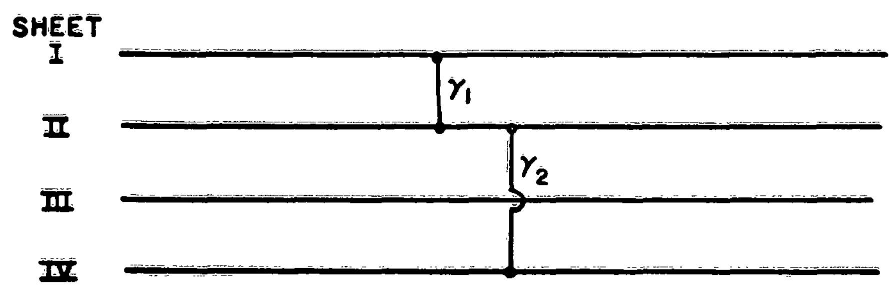

Figure 9.2 - The Four-sheeted Riemann Surface in the β -Plane

Now we focus our attention on the following. As we remarked earlier, the integrands under consideration are four-valued since they contain the radicals of σ_1 and σ_2. The original path of integration passes over Sheet I of the four-sheeted Riemann surface and can be deformed into the path of the steepest descents Γ only when at least the beginning and the end of it lie on this sheet. In the case when the angle θ does not exceed one of the angles of the total internal reflection θ_{s_1} or θ_{s_2}, then in the complex β-plane we have a picture as shown in Fig. 9.1. Here the path of the steepest descents crosses one or both branch cuts twice and thus ends on Sheet I. Transition from path Γ_1 to the path Γ is then accomplished without any complications. Except for the part shown dotted in Fig. 9.1, the path Γ will lie on the Sheet I. In this case the only contribution to the field integrals will be from the saddle point.

The situation is quite different when the angle θ exceeds the angle of the total internal reflection of one or both waves. Suppose that $\theta_{s_1} < \theta < \theta_{s_2}$. Then the path Γ will cross the branch cut γ_1 only once and the branch cut γ_2 twice. Thus, the path Γ will cross from Sheet I onto Sheet II and then IV and then back to II which is inadmissible. When $\theta > \theta_{s_2} > \theta_{s_1}$ the path Γ will cross each cut γ_1 and γ_2 only once. Thus, the path Γ will cross from Sheet I onto Sheet II and then onto Sheet IV which again is inadmissible. We can, however, construct a more complicated path of integration, supplementing the path of the steepest descents Γ by a contour encompassing each cut in such a way that the beginning and the end of the more complicated path will lie on Sheet I. This situation is similar to that in Section 4.2 only that here two branch cuts are involved.

To this end we propose the contour shown in Fig. 9.3. For $\theta > 0$ this new path of integration runs along the path of the steepest descents, Γ, then along the borders of the cuts, intersecting each branch cut once thus entering

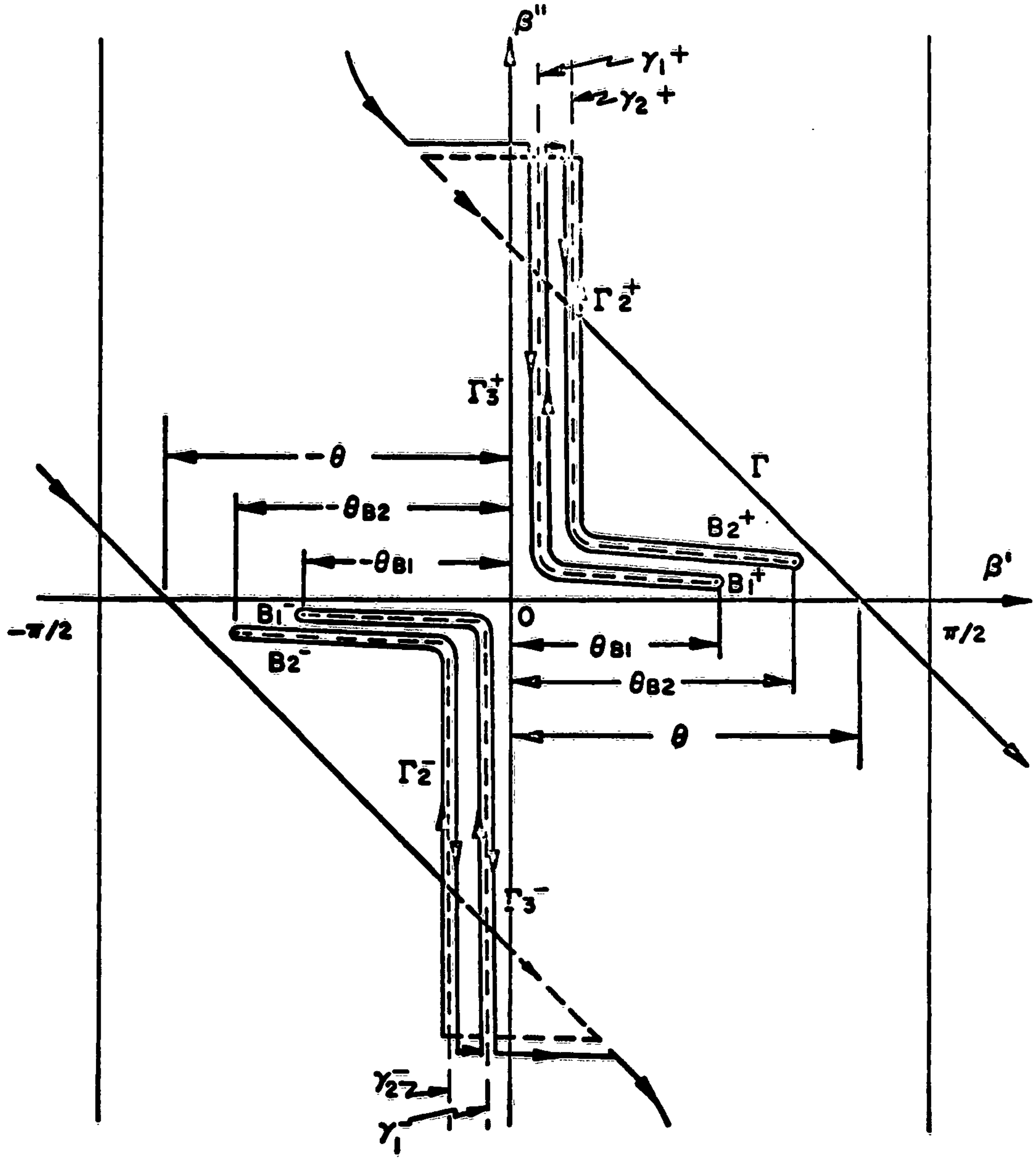

Figure 9.3 - Integration Paths Along the Branch Cuts

Sheet IV. It then continues along the path Γ and intersecting each branch cut once more it comes back on Sheet I where it continues along the path Γ. For $\theta < 0$ the situation is analogous.

As a consequence the complete expression for any of the field integrals will consist of three parts and can be written

$$W = W^{(s)} + W^{(B_1)}u(\theta - \theta_{B_1}) + W^{(B_2)}u(\theta - \theta_{B_2}) \tag{9.10}$$

where $W^{(s)}$ will denote the contribution from the saddle point and $W^{(B_1)}$ and $W^{(B_2)}$ will denote the contributions from the borders of the cut associated with branch point B_1 and B_2 respectively.

<u>9.2 a Formulation of the contribution from the saddle point</u>—As we remarked earlier, each of the field integrals under consideration can be written in the form

$$W = \int_{\Gamma_1} F(\beta)\, e^{ik_0\rho\cos(\beta-\theta)} d\beta . \tag{9.11}$$

The saddle point of the integrand occurs where the derivative of the exponent vanishes, i.e., at $\beta = \theta$. The leading term of the asymptotic expansion for (9.11) will then be given by

$$W^{(s)} \sim \sqrt{\frac{2\pi}{k_0\rho}}\; F(\theta)\, e^{i(k_0\rho - \pi/4)}. \tag{9.12}$$

<u>9.2 b Formulation of the contributions from the branch cuts</u>—First consider the contribution to the integral (9.11) due to the integration along the borders of the cut corresponding to the branch point B_1. We write

$$W^{(B_1)} = \int_{\Gamma_3^+} F(\beta)\, e^{ik_0\rho\cos(\beta-\theta)} d\beta \tag{9.13}$$

and integrate formally on both sides of the cut getting

$$W^{(B_1^+)} = \int_{i\infty}^{B_1^+} F_+^{(1)}(\beta) e^{ik_0\rho\cos(\beta-\theta)} d\beta + \int_{B_1^+}^{i\infty} F_-(\beta) e^{ik_0\rho\cos(\beta-\theta)} d\beta$$

$$= \int_{B_1^+}^{i\infty} R^{(1)}(\beta) e^{ik_0\rho\cos(\beta-\theta)} d\beta \tag{9.14}$$

where

$$R^{(1)}(\beta) = F_-^{(1)}(\beta) - F_+^{(1)}(\beta) \tag{9.15}$$

and $F_+^{(1)}(\beta)$ denotes the value of the function $F(\beta)$ on the left side of the cut $\gamma_1{}^+$ and $F_-^{(1)}(\beta)$ is the value of the same function on the right side. Next we deform the path of integration to the path of the steepest descents through the point B_1^+ . The procedure is identical to that in Section 4.2 b and we write

$$W^{(B_1^+)} = i e^{ik_0\rho\cos(\theta-\theta_{B_1})} \int_0^{\infty} x R^{(1)}(\beta) e^{-k_0\rho\sin(\theta-\theta_{B_1})x^2/2} dx \tag{9.16}$$

$$\beta = \theta_{B_1} + \frac{ix^2}{2} .$$

By analogy we can write similar expressions for the contribution from the borders of the cut $\gamma_2{}^+$ associated with the branch-point B_2^+ . We have

$$W^{(B_2)} = i e^{ik_0\rho\cos(\theta-\theta_{B_2})} \int_0^{\infty} x R^{(2)}(\beta) e^{-k_0\rho\sin(\theta-\theta_{B_2})x^2/2} dx \tag{9.17}$$

$$\beta = \theta_{B_2} + \frac{ix^2}{2} .$$

Now we find the contributions from the branch cuts when the angle $\theta < 0$. Integrating along the borders of the cut associated with the branch-point B_1^-, we obtain

$$W^{(B_1^-)} = \int_{-i\infty}^{B_1^-} F_-^{(1)}(\beta)\, e^{ik_0\rho\cos(\beta-\theta)}\, d\beta + \int_{B_1^-}^{-i\infty} F_+^{(1)}(\beta)\, e^{ik_0\rho\cos(\beta-\theta)}\, d\beta$$

$$= -\int_{B_1^-}^{-i\infty} R^{(1)}(\beta)\, e^{ik_0\rho\cos(\beta-\theta)}\, d\beta \qquad (9.18)$$

where $R^{(1)}(\beta)$ is given by (9.15). Following the procedure of Section 4.2, we deform the path of integration to a path of steepest descents through the point B_1^- and obtain as a result

$$W^{(B_1^-)} = ie^{ik_0\rho\cos(|\theta|-\theta_{B1})} \int_0^\infty x\, R^{(1)}(-\beta)\, e^{-k_0\rho\sin(|\theta|-\theta_{B1})x^2/2}\, dx \ . \qquad (9.19)$$

By analogy the contribution from the branch cut associated with the branch-point B_2^- can be written at once. We have

$$W^{(B_2^-)} = ie^{ik_0\rho\cos(|\theta|-\theta_{B2})} \int_0^\infty x\, R^{(2)}(-\beta)\, e^{-k_0\rho\sin(|\theta|-\theta_{B2})x^2/2}\, dx \ . \qquad (9.20)$$

We can combine the results of (9.16) and (9.19) as well as (9.17) and (9.20) and write

$$W^{(B_1)}_{\theta\gtrless 0} = ie^{ik_0\rho\cos(|\theta|-\theta_{B1})} \int_0^\infty x\, R^{(1)}(\pm\beta)\, e^{-k_0\rho\sin(|\theta|-\theta_{B1})x^2/2}\, dx \qquad (9.21a)$$

$$W^{(B_2)}_{\theta\gtrless 0} = ie^{ik_0\rho\cos(|\theta|-\theta_{B2})} \int_0^\infty x\, R^{(2)}(\pm\beta)\, e^{-k_0\rho\sin(|\theta|-\theta_{B2})x^2/2}\, dx \qquad (9.21b)$$

9.3 EVALUATION OF THE FIELD INTEGRALS AT THE SADDLE-POINT

In this section we shall apply the result of the formulation (9.12) to find the saddle-point contribution to the field integrals. Because of the complexity of the results we shall only find the leading term representing the radiation field.

9.3 a The field component $E_{yo}^{(s)}$—Application of (9.12) to the integral expression for $E_{yo}^{(s)}$ in (9.1a) gives

$$E_{yo}^{(s)} \sim -\frac{\omega\mu_o I}{\sqrt{2\pi}} F_{yo}(\theta)\cos\theta \frac{e^{i(k_o\rho-\pi/4)}}{\sqrt{k_o\rho}} \tag{9.22}$$

where

$$F_{yo}(\theta)=\frac{P_{E1}(\theta)\left[\sigma_2(\theta)+\zeta\cos\theta\right]e^{ik_oh\sigma_1(\theta)}-P_{E2}(\theta)\left[\sigma_1(\theta)+\zeta\cos\theta\right]e^{ik_oh\sigma_2(\theta)}}{\mathcal{N}_E(\theta)} \tag{9.23a}$$

$$\mathcal{N}_E(\theta)=P_{E1}(\theta)\left[\sigma_1(\theta)+\cos\theta\right]\left[\sigma_2(\theta)+\zeta\cos\theta\right]-P_{E2}(\theta)\left[\sigma_2(\theta)+\cos\theta\right]\cdot\left[\sigma_1(\theta)+\zeta\cos\theta\right] \tag{9.23b}$$

$$P_{E1,2}(\theta) = R(\theta) \mp \sqrt{R^2(\theta) + \zeta\eta^2\sin^2\theta} \tag{9.23c}$$

$$\sigma_{1,2}(\theta) = \sqrt{\frac{\zeta(\epsilon-\sin^2\theta) - P_{E1,2}(\theta)}{\epsilon}} \tag{9.23d}$$

$$R(\theta) = \frac{\epsilon\zeta - \epsilon^2 + \eta^2 + (\epsilon-\zeta)\sin\theta}{2}\,. \tag{9.23e}$$

9.3 b The field component $E_{\theta o}^{(s)}$—Application of (9.12) to the integral expression for $E_{\theta o}^{(s)}$ in (9.1b) gives

$$E_{\theta o}^{(s)} \sim \frac{i\eta\zeta\omega\mu_o I}{\epsilon\sqrt{2\pi}} F_{\theta o}(\theta)\sin\theta\cos\theta \frac{e^{i(k_o\rho-\pi/4)}}{\sqrt{k_o\rho}} \tag{9.24}$$

where

$$F_{\theta o} = \frac{\left[\sigma_2(\theta)+\cos\theta\right]e^{ik_oh\sigma_1(\theta)} - \left[\sigma_1(\theta)+\cos\theta\right]e^{ik_oh\sigma_2(\theta)}}{\mathcal{N}_E(\theta)} \tag{9.25}$$

and $\sigma_1(\theta)$, $\sigma_2(\theta)$, and $\mathcal{N}_E(\theta)$ were defined previously.

<u>9.3 c The field component $E_{\rho o}^{(s)}$</u>—As can be noted from the integral representation of $E_{\rho o}^{(s)}$ in equation (9.1c), the integrand is zero at the saddle point $\beta = \theta$. Thus,

$$E_{\rho o}^{(s)} = 0 \tag{9.26}$$

9.4 EVALUATION OF THE FIELD INTEGRALS ALONG THE BRANCH CUTS

In Section 9.2 we have formulated in general terms the contribution to the field integrals due to the integration along the borders of the branch cuts. In this section we shall apply this formulation to find the branch-cut contributions to the field integrals. These branch-cut contributions are the sources of the lateral waves as we remarked earlier.

<u>9.4a The field components $E_{yo}^{(B_1)}$ and $E_{yo}^{(B_2)}$</u>—We apply (9.21a) to the integral representation of E_{yo} in equation (9.1a) and noting that the integrand is an even function of β, we obtain

$$E_{yo}^{(B_1)} = -\frac{i\omega\mu_0 I e^{ik_0\rho\cos(|\theta|-\theta_{B_1})}}{2\pi}\int_0^{\infty} x R_{yo}^{(1)}(\beta)\, e^{-k_0\rho\sin(|\theta|-\theta_{B_1})x^2/2}\,dx \tag{9.27}$$

where by (9.16) and (9.1a)

$$R_{yo}^{(1)} = \cos\beta\left\{\frac{P_E(\sigma_1)(\sigma_2+\zeta\cos\beta)e^{-ik_0h\sigma_1} - P_E(\sigma_2)(-\sigma_1+\zeta\cos\beta)e^{ik_0h\sigma_2}}{(-\sigma_1+\cos\beta)(\sigma_2+\zeta\cos\beta)P_E(\sigma_1)-(\sigma_2+\cos\beta)(-\sigma_1+\zeta\cos\beta)P_E(\sigma_2)}\right.$$
$$\left. - \frac{P_E(\sigma_1)(\sigma_2+\zeta\cos\beta)e^{ik_0h\sigma_1} - P_E(\sigma_2)(\sigma_1+\zeta\cos\beta)e^{ik_0h\sigma_2}}{(\sigma_1+\cos\beta)(\sigma_2+\zeta\cos\beta)P_E(\sigma_1)-(\sigma_2+\cos\beta)(\sigma_1+\zeta\cos\beta)P_E(\sigma_2)}\right\} \tag{9.28}$$

where

$$\beta = \theta_{B_1} + \frac{i\,x^2}{2} \tag{9.29a}$$

$$\sin\theta_{B_1} = \sqrt{\epsilon+\eta} \tag{9.29b}$$

Now we must express $R_{yo}^{(1)}$ in terms of x. To accomplish this we set $\beta = \theta_{B1}$ everywhere except in σ_1 which becomes zero when $\beta = \theta_{B1}$. We evaluate σ_1 then to the next higher approximation. To this end we note

$$\sin^2\beta \sim \sin^2\theta_{B1} + ix^2 \sin\theta_{B1} \cos\theta_{B1} \tag{9.30}$$

and we find σ_1 to be

$$\sigma_1 = - e^{-i\pi/4} \sqrt{\frac{\zeta \sin 2\theta_{B1}}{\zeta + \epsilon + \eta}}\, x \,. \tag{9.31}$$

Furthermore, we note

$$\begin{aligned} \sigma_2^2 &= - \frac{\eta}{\epsilon} (\epsilon + \eta + \zeta) \\ P_E(\sigma_1) &= - \zeta\eta \\ P_E(\sigma_2) &= \eta(\epsilon + \eta). \end{aligned} \tag{9.32}$$

Substituting the above results into (9.28) and carrying out the necessary algebraic operations, we find the leading term

$$R_{yo}^{(1)} = 2\, C_{yo}^{(1)}\, e^{-i\pi/4} x \tag{9.33}$$

where

$$C_{yo}^{(1)} = \frac{\zeta\cos\theta_{B1}\sqrt{\zeta\sin 2\theta_{B1}}}{b_{E1}^2\sqrt{\epsilon+\eta+\zeta}} \left[\zeta\cos\theta_{B1} + \sqrt{-\kappa(\epsilon+\eta+\zeta)}\right] \cdot \tag{9.34a}$$

$$\cdot \left[ik_0 h b_{B1} - a_{B1} + (1-\zeta)(\epsilon+\eta) e^{ik_0 h\sqrt{-\kappa(\epsilon+\eta+\zeta)}} \cdot \cos\theta_{B1}\right]$$

and

$$a_{E1} = (\epsilon+\eta+\zeta)\cdot\sqrt{-\frac{\eta}{\epsilon}(\epsilon+\eta+\zeta)} + (\epsilon+\eta+\zeta^2)\cdot\sqrt{1-(\epsilon+\eta)} \tag{9.34b}$$

$$b_{E1} = \zeta\sqrt{1-(\epsilon+\eta)}\left[(1+\epsilon+\eta)\sqrt{-\frac{\eta}{\epsilon}(\epsilon+\eta+\zeta)} + (\epsilon+\eta+\zeta)\sqrt{1-(\epsilon+\eta)}\right]. \tag{9.34c}$$

Performing the integration where we use the results of (4.31b) we obtain, finally,

$$E_{yo}^{(B1)} = - \frac{i\omega\mu_0 I}{\sqrt{2\pi}}\, C_{yo}^{(1)}\, \frac{e^{ik_0\rho\cos(|\theta|-\theta_{B1})}\, e^{-i\pi/4}}{\left[k_0\rho\sin(|\theta| - \theta_{B1})\right]^{3/2}} \,. \tag{9.35}$$

Now we evaluate the same integral along the borders of the second cut. We write

$$E_{yo}^{(\theta_2)} = \frac{-i\omega\mu_0 I e^{ik_0\rho\cos(|\theta|-\theta_{B2})}}{2\pi}\int_0^\infty x R_{yo}^{(2)}(\beta)\, e^{-k_0\rho\sin(|\theta|-\theta_{B2})x^2/2}\,dx \tag{9.36}$$

where

$$R_{yo}^{(2)} = \cos\beta\left\{\frac{P_E(\sigma_1)(-\sigma_2+\zeta\cos\beta)e^{ik_0h\sigma_1} - P_E(\sigma_2)(\sigma_1+\zeta\cos\beta)e^{-ik_0h\sigma_2}}{(\sigma_1+\cos\beta)(-\sigma_2+\zeta\cos\beta)P_E(\sigma_1)-(-\sigma_2+\cos\beta)(\sigma_1+\zeta\cos\beta)P_E(\sigma_2)}\right.$$
$$\left. - \frac{P_E(\sigma_1)(\sigma_2+\zeta\cos\beta)e^{ik_0h\sigma_1} - P_E(\sigma_2)(\sigma_1+\zeta\cos\beta)e^{ik_0h\sigma_2}}{(\sigma_1+\cos\beta)(\sigma_2+\zeta\cos\beta)P_E(\sigma_1)-(\sigma_2+\cos\beta)(\sigma_1+\zeta\cos\beta)P_E(\sigma_2)}\right\} \tag{9.37}$$

where

$$\beta = \theta_{B2} + \frac{ix^2}{2} \tag{9.38a}$$

$$\sin\theta_{B2} = \sqrt{\epsilon-\eta}\,. \tag{9.38b}$$

To express $R_{yo}^{(2)}$ in terms of x we set $\beta = \theta_{B2}$ everywhere except in σ_2 which becomes zero when $\beta = \theta_{B2}$. We evaluate σ_2 then to the next higher approximation, getting

$$\sigma_2 = -e^{-i\pi/4}\sqrt{\frac{\zeta\sin 2\theta_{B2}}{\zeta+\epsilon-\eta}}\; x \tag{9.39}$$

Furthermore, we note

$$\begin{aligned}\sigma_1^2 &= \frac{\eta}{\epsilon}(\zeta+\epsilon-\eta)\\ P_E(\sigma_1) &= -\eta(\epsilon-\eta)\\ P_E(\sigma_2) &= \zeta\eta.\end{aligned} \tag{9.40}$$

Substituting the above results into (9.37) we obtain for the leading term

$$R_{yo}^{(2)} = 2C_{yo}^{(2)} e^{-i\pi/4} x \tag{9.41}$$

where

$$C_{yo}^{(2)} = \frac{\zeta\cos\theta_{B2}\sqrt{\zeta\sin 2\theta_{B2}}}{b_{E2}^{2}\sqrt{\zeta+\epsilon-\eta}}\left[\zeta\cos\theta_{B2}+\sqrt{\kappa(\zeta+\epsilon-\eta)}\right] \tag{9.42}$$

$$\cdot\left[ik_o h b_{E2} - a_{E2} + (1-\zeta)(\epsilon-\eta)e^{ik_o h\sqrt{\kappa(\zeta+\epsilon-\eta)}}\cdot\cos\theta_{E2}\right]$$

and

$$a_{E2} = (\zeta+\epsilon-\eta)\sqrt{\tfrac{2}{\epsilon}(\zeta+\epsilon-\eta)} + (\zeta^2+\epsilon-\eta)\sqrt{1-(\epsilon-\eta)} \tag{9.43a}$$

$$b_{E2} = \zeta\sqrt{1-(\epsilon-\eta)}\left[(1+\epsilon-\eta)\sqrt{\tfrac{2}{\epsilon}(\zeta+\epsilon-\eta)}+(\zeta+\epsilon-\eta)\sqrt{1-(\epsilon-\eta)}\right] \tag{9.43b}$$

Performing the integration where we use the results of (4.31b) we obtain, finally,

$$E_{yo}^{(\theta_2)} = -\frac{i\omega\mu_o I}{\sqrt{2\pi}}C_{yo}^{(2)}\frac{e^{ik_o\rho\cos(|\theta|-\theta_{B2})}e^{-i\pi/4}}{\left[k_o\rho\sin(|\theta|-\theta_{B2})\right]^{3/2}}. \tag{9.44}$$

9.4 b The field components $E_{\theta o}^{(\theta_1)}$ and $E_{\theta o}^{(\theta_2)}$ — We apply (9.21a) to the integral representation of $E_{\theta o}$ in equation (9.1b)

$$E_{\theta o}^{(\theta_1)} = \frac{i\eta\zeta\omega\mu_o I}{2\pi k_o\epsilon}\partial_x\left\{e^{ik_o\rho\cos(|\theta|-\theta_{B1})}\int_0^\infty R_{\theta o}^{(1)}(\pm\beta)e^{-k_o\rho\sin(|\theta|-\theta_{B1})x^2/2}dx\right\} \tag{9.45}$$

where

$$R_{\theta o}^{(1)}(\pm\beta) = \cos\beta\cos(\beta-|\theta|)\left\{\frac{(\sigma_2+\cos\beta)e^{-ik_o h\sigma_1}+(\sigma_1-\cos\beta)e^{ik_o h\sigma_2}}{(-\sigma_1+\cos\beta)(\sigma_2+\zeta\cos\beta)P_E(\sigma_1)-(\sigma_2+\cos\beta)(-\sigma_1+\zeta\cos\beta)P_E(\sigma_2)}\right.$$

$$\left.-\frac{(\sigma_2+\cos\beta)e^{ik_o h\sigma_1}-(\sigma_1+\cos\beta)e^{ik_o h\sigma_2}}{(\sigma_1+\cos\beta)(\sigma_2+\zeta\cos\beta)P_E(\sigma_1)-(\sigma_2+\cos\beta)(\sigma_1+\zeta\cos\beta)P_E(\sigma_2)}\right\} \tag{9.46}$$

Substituting the expressions for σ_2, $P_E(\sigma_1)$ and $P_E(\sigma_2)$ from (9.32) we obtain for the leading term

$$R_{\theta o}^{(1)} = \frac{-2C_{\theta o}^{(1)}\cos(|\theta|-\theta_{B1})e^{-i\pi/4}}{\eta\sqrt{\epsilon+\eta}}\,x \tag{9.47}$$

where

$$C^{(1)}_{\theta o} = \frac{\sqrt{\zeta}\,[\sin 2\theta_{B1}]^{3/2}(\cos\theta_{B1} + \sqrt{-\kappa(\zeta+\epsilon+\eta)})}{2 b^2_{B1}\sqrt{\zeta+\epsilon+\eta}} \tag{9.48}$$

$$\cdot \left[i k_o h b_{B1} - a_{B1} + (1-\zeta)(\epsilon+\eta)\cos\theta_{B1}\, e^{i k_o h \sqrt{-\kappa(\zeta+\epsilon+\eta)}} \right]$$

and where a_{B1} is given by (9.34b) and b_{B1} by (9.34c). Performing the integration where we use the result of (4.31b) we obtain

$$E^{(B_1)}_{\theta o} = \frac{-i\zeta\omega\mu_o I\, e^{-i\pi/4}}{\sqrt{2\pi}\, k_o\epsilon\sqrt{\epsilon+\eta}}\, C^{(1)}_{\theta o}\, \partial_x \left\{ \frac{\cos(|\theta|-\theta_{B1})\, e^{i k_o\rho\cos(|\theta|-\theta_{B1})}}{[k_o\rho\sin(|\theta|-\theta_{B1})]^{3/2}} \right\} \tag{9.49}$$

To perform the differentiation we use (8.35) and note

$$\partial_x \underset{\theta \gtrless 0}{=} \pm i\, k_o \sin\theta_{B1} = \pm i\, k_o \sqrt{\epsilon+\eta} \tag{9.50}$$

for the leading term. Thus, finally

$$E^{(B_1)}_{\theta o} \underset{\theta \gtrless 0}{=} \pm \frac{\zeta\omega\mu_o I}{\epsilon\sqrt{2\pi}}\, C^{(1)}_{\theta o}\, \frac{\cos(|\theta|-\theta_{B1})\, e^{i k_o\rho\cos(|\theta|-\theta_{B1})}\, e^{-i\pi/4}}{[k_o\rho\sin(|\theta|-\theta_{B1})]^{3/2}} \tag{9.51}$$

Now we evaluate the same integral along the borders of the second branch cut. We write

$$E^{(B_2)}_{\theta o} = \frac{i\eta\zeta\omega\mu_o I}{2\pi k_o\epsilon}\, \partial_x \left\{ e^{i k_o\rho\cos(|\theta|-\theta_{B2})} \int_0^\infty R^{(2)}_{\theta o}(\pm\beta)\, e^{-k_o\rho\sin(|\theta|-\theta_{B2})x^2/2}\, dx \right\} \tag{9.52}$$

where

$$R^{(2)}_{\theta o}(\pm\beta) = \cos\beta\cos(\beta-|\theta|) \left\{ \frac{(-\sigma_2+\cos\beta)e^{i k_o h\sigma_1} - (\sigma_1+\cos\beta)e^{-i k_o h\sigma_2}}{(\sigma_1+\cos\beta)(-\sigma_2+\zeta\cos\beta)P_E(\sigma_1) - (-\sigma_2+\cos\beta)(\sigma_1+\zeta\cos\beta)P_E(\sigma_2)} \right.$$

$$\left. - \frac{(\sigma_2+\cos\beta)e^{i k_o h\sigma_1} - (\sigma_1+\cos\beta)e^{i k_o h\sigma_2}}{(\sigma_1+\cos\beta)(\sigma_2+\zeta\cos\beta)P_E(\sigma_1) - (\sigma_2+\cos\beta)(\sigma_1+\zeta\cos\beta)P_E(\sigma_2)} \right\} \tag{9.53}$$

Substituting for σ_1, $P_E(\sigma_1)$, and $P_E(\sigma_2)$, the expression found in (9.40), we obtain for the leading term

$$E_{\theta o}^{(2)} = + \frac{2\, C_{\theta o}^{(2)} \cos(|\theta| - \theta_{B2}) e^{-i\pi/4}}{\eta\sqrt{\epsilon - \eta}} \; x \tag{9.54}$$

where

$$C_{\theta o}^{(2)} = \frac{\sqrt{\zeta}\,[\sin 2\theta_{B2}]^{3/2}(\cos\theta_{B2} + \sqrt{\kappa(\zeta+\epsilon-\eta)})}{2 b_{B2}^2 \sqrt{\zeta+\epsilon-\eta}} \tag{9.55}$$

$$\cdot\left[i k_o h\, b_{B2} - a_{B2} + (1-\zeta)(\epsilon-\eta)\cos\theta_{B2}\, e^{i k_o h \sqrt{\kappa(\zeta+\epsilon-\eta)}} \right]$$

and a_{B2} is given by (9.43a) and b_{B2} by (9.43b). Performing the integration where we use the results of (4.31b) we obtain

$$E_{\theta o}^{(B_2)} = + \frac{i\zeta\omega\mu_o I e^{-i\pi/4}}{\sqrt{2\pi}\, k_o \epsilon\sqrt{\epsilon-\eta}}\, C_{\theta o}^{(2)}\, \partial_x\left\{ \frac{\cos(|\theta|-\theta_{B2})\, e^{i k_o \rho \cos(|\theta|-\theta_{B2})}}{[k_o\rho\sin(|\theta|-\theta_{B2})]^{3/2}} \right\}. \tag{9.56}$$

To carry out the indicated differentiation we note

$$\partial_{x}\Big|_{\theta \gtrless 0} = \pm i\, k_o \sin\theta_{B2} = \pm i\, k_o \sqrt{\epsilon - \eta} \tag{9.57}$$

for the leading term. Thus, we obtain finally

$$E_{\theta o}^{(B_2)}\Big|_{\theta \gtrless 0} = \mp \frac{\zeta\omega\mu_o I\, C_{\theta o}^{(2)}}{\sqrt{2\pi}\,\epsilon}\; \frac{\cos(|\theta| - \theta_{B2})\, e^{i k_o \rho\cos(|\theta|-\theta_{B2})}\, e^{-i\pi/4}}{[k_o\rho\sin(|\theta| - \theta_{B2})]^{3/2}}. \tag{9.58}$$

<u>9.4 c The field components $E_{\rho o}^{(B_1)}$ and $E_{\rho o}^{(B_2)}$</u> — To find $E_{\rho o}^{(B_1)}$ and $E_{\rho o}^{(B_2)}$ we could proceed as before. This is, however, not necessary if we notice the similarity of the integrands corresponding to $E_{\theta o}$ and $E_{\rho o}$ in equations (9.1b) and (9.1c). Thus, $E_{\rho o}^{(B_1)}$ can be obtained from $E_{\theta o}^{(B_1)}$ by putting $\pm\sin(|\theta| - \theta_{B1})$ in place of $\cos(|\theta| - \theta_{B1})$ in the numerator. Similarly $E_{\rho o}^{(B_2)}$ can be obtained from $E_{\theta o}^{(B_2)}$ by putting $\pm\sin(|\theta| - \theta_{B2})$ in place of $\cos(|\theta| - \theta_{B2})$ in the

numerator. Thus, we can write immediately

$$E_{\rho o}^{(B_1)} = \frac{+\zeta\omega\mu_o I C_{\theta o}^{(1)}}{\sqrt{2\pi}\,\epsilon} \cdot \frac{\sin(|\theta| - \theta_{B_1})\, e^{ik_o\rho\cos(|\theta| - \theta_{B_1})}\; e^{-i\pi/4}}{[k_o\rho\sin(|\theta| - \theta_{B_1})]^{3/2}} \tag{9.59}$$

and

$$E_{\rho o}^{(B_2)} = \frac{-\zeta\omega\mu_o I C_{\theta o}^{(2)}}{\sqrt{2\pi}\,\epsilon} \quad \frac{\sin(|\theta| - \theta_{B_2})\, e^{ik_o\rho\cos(|\theta| - \theta_{B_2})}\; e^{-i\pi/4}}{[k_o\rho\sin(|\theta| - \theta_{B_2})]^{3/2}} \tag{9.60}$$

<u>9.4 d Physical interpretation of the lateral waves</u>--The contribution of the branch-cut integration found above has a well defined meaning in the physics of wave propagation in layered media (5,p.270). It is the mathematical equivalent of the physical phenomenon of lateral waves whose historical account we have given in Section 1.2. Here we shall attempt to explain physically the mechanism of the lateral wave formation on magnetoplasma interfaces.

To this end we consider the geometry of the Fig. 9.4. We note

$$x = d_1 + g_1 \sin\theta_{B_1} \tag{9.61a}$$

$$z = g_1 \cos\theta_{B_1} \tag{9.61b}$$

as a consequence of which we obtain

$$k_o\rho\cos(\theta - \theta_{B_1}) = k_o\sqrt{\epsilon+\eta}\; d_1 + k_o g_1 \tag{9.62a}$$

$$\rho\sin(\theta - \theta_{B_1}) = \sqrt{1-(\epsilon+\eta)}\; d_1 \tag{9.62b}$$

Using the above results we can recast the field components due to the branch-cut integration in the following form:

$$E_o^{(B_1)} \sim \frac{e^{i(k_o\sqrt{\epsilon+\eta}\,d_1 + k_o g_1)}}{d_1^{3/2}} \tag{9.63a}$$

$$E_o^{(B_2)} \sim \frac{e^{i(k_o\sqrt{\epsilon-\eta}\,d_2 + k_o g_2)}}{d_2^{3/2}} \tag{9.63b}$$

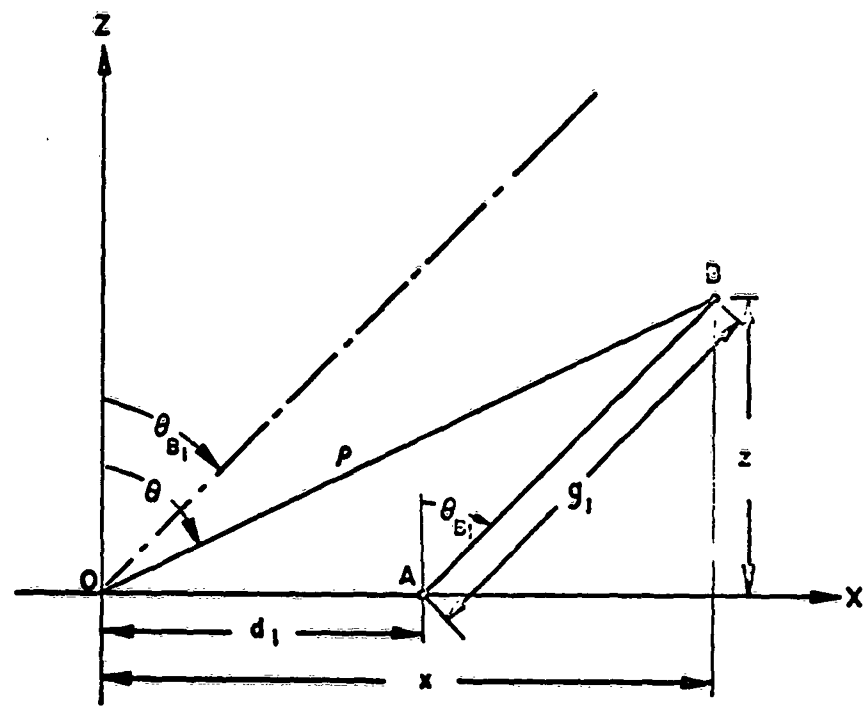

Figure 9.4 - Concerning the Geometry of a Lateral Wave

In the above we recognise that $k_o\sqrt{\epsilon+\eta}$ and $k_o\sqrt{\epsilon-\eta}$ are the propagation factors of plane ordinary and extraordinary rays respectively in a magnetoplasma when the direction of propagation coincides with the direction of the steady magnetic field. Thus, the total phase of each one of the lateral waves consists of two parts. The part for example, $k_o\sqrt{\epsilon+\eta}\,d_1$, tells us that the wave travels along the interface in the magnetoplasma through a distance d_1 with the phase velocity of $c/\sqrt{\epsilon+\eta}$. At point A, Fig. 9.4, the wave changes its direction and travels along the path A - B with the phase velocity c, which is the phase velocity of free space. The other lateral wave associated with the second branch cut, B_2, follows a similar pattern.

The complete picture of both waves is shown in Fig. 9.5 where we assumed for convenience that the source is located at the interface. The ray OB is the direct ray from the source representing the radiation field as found from the saddle-point integration. The lateral wave represented by the ray OA propagates along the interface with the phase velocity of the ordinary wave in the magnetoplasma. Since this velocity is different than the one "allowed" by the free space, there is a disturbance created along the interface in the air as a result of which a new wave is produced represented by the ray AB. Since the spatial period of this disturbance along the interface is equal to $\lambda_1^{(o)}$, the wave-length of the ordinary wave in the magnetoplasma, the wave-length in the air can "fit" this period only if the angle θ_{s1} between the normal to the boundary and the direction of propagation is such that the wave-length in the air $\lambda_o = \lambda_1^{(o)}\sin\theta_{s1}$. This is just the direction of the lateral wave. The same argument holds for the second lateral wave represented by the ray OA. The resultant field at a point B is a superposition of three waves as we concluded earlier in equation (9.10).

The diagram of Fig. 9.5 depicts the situation when $p^2 < 1 \pm \sigma$, i.e., when

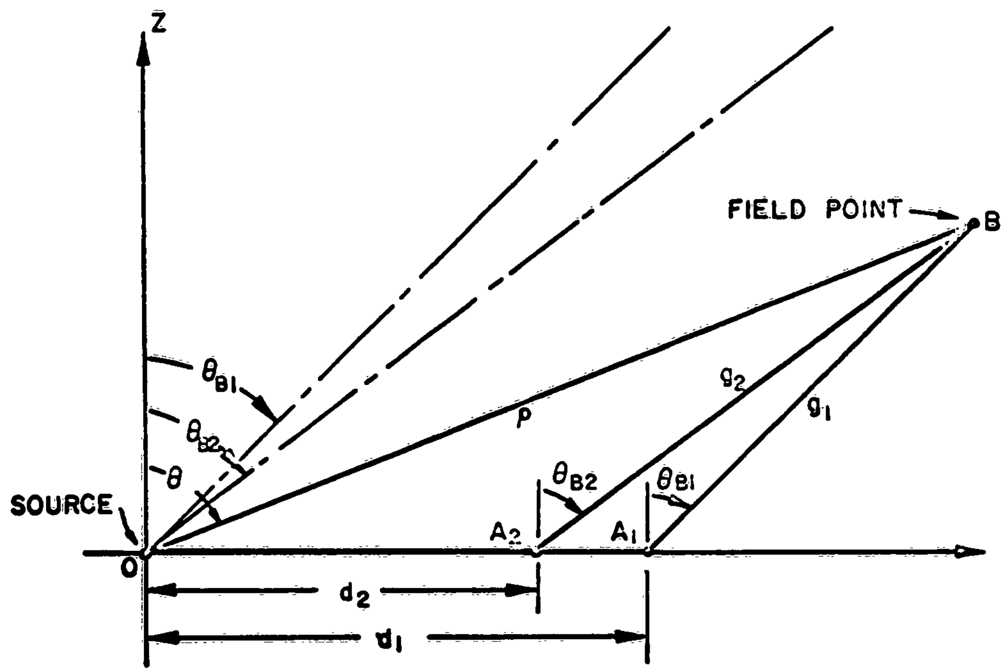

Figure 9.5 - Geometry of Two Lateral Waves When $p^2 < 1 \pm \sigma$

$$\left(\frac{\omega_p}{\omega}\right)^2 < 1 \pm \frac{\omega_H}{\omega} \quad . \tag{9.64}$$

If the above inequality is not satisfied, one or both of the lateral waves will disappear. Physically this occurs when either θ_{s_1}, or θ_{s_2}, or both are outside the range of interest 0 to $\pi/2$.

9.5 THE POWER FLOW

In the preceding section we found the electric field components in the air. The form of these components tell a great deal about the physics of the problem. However, still a better insight into the problem can be obtained by considering the direction and magnitude of the energy carried by the wave represented by these field components.

Now the time averaged energy density carried by an electromagnetic wave is given by Poynting vector

$$\vec{S} = \frac{1}{2} \operatorname{Re} \left\{ \vec{E} \times \vec{H}^* \right\} \tag{9.65}$$

which can be written in component form

$$S_\rho = \frac{1}{2} \operatorname{Re} \left(E_\varphi H_y^* - E_y H_\varphi^* \right) \tag{9.66a}$$

$$S_\varphi = \frac{1}{2} \operatorname{Re} \left(E_y H_\rho^* - E_\rho H_y^* \right) \tag{9.66b}$$

$$S_y = \frac{1}{2} \operatorname{Re} \left(E_\rho H_\varphi^* - E_\varphi H_\rho^* \right) \tag{9.66c}$$

Thus, before we determine the components of the Poynting vector we first must find the components of the magnetic field vector.

<u>9.5 a The magnetic field vector</u>—The magnetic field vector in the air is obtainable from

$$\vec{H}_o = \frac{1}{i\omega\mu_o} \vec{\nabla} \times \vec{E}_o \tag{9.67}$$

In what follows we shall only be concerned with the radiation field since the lateral field is of second order. Thus, in the radiation field $E_{\rho o} = 0$ and moreover to the first order

$$\begin{aligned} \partial_\rho &= ik_o \\ \partial_\theta &= 0 \end{aligned} \tag{9.68}$$

which enables us to find the components of the magnetic field vector as follows:

$$H_{\theta o} = -\frac{1}{Z_o}\ E_{yo} \tag{9.69a}$$

$$H_{yo} = \frac{1}{Z_o}\ E_{\theta o} \tag{9.69b}$$

where Z_o is the free-space impedance.

9.5 b The components of the Poynting vector—Using the results of (9.69) together with (9.66) we obtain only one component of $\vec{S}$

$$S_{\rho o} = \frac{1}{2\ Z_o}\ \left(|E_{yo}|^2 + |E_{\theta o}|^2 \right) \tag{9.70}$$

Now using the results of (9.22) and (9.24) we obtain for the first order term:

$$S = \frac{\omega\mu_o I^2 \cos^2\theta}{4\pi\rho}\ \left(|F_{yo}|^2 + \frac{\eta^2 \zeta^2}{\epsilon^2} \sin^2\theta\ |F_{\theta o}|^2 \right) \tag{9.71}$$

where F_{yo} is given by (9.23) and $F_{\theta o}$ by (9.25). We note that the essential difference between the above result and the result obtained from a corresponding isotropic case is the presence of the second term.

9.5 c Numerical example—The power pattern represented by equation (9.71) was computed and plotted in Figures 9.7 and 9.8 for an electric current line source situated at the lower edge of the ionosphere. For the purpose of this example the lower edge of the ionosphere was assumed to be sharply bounded, having an electron concentration N = 750 electrons per cubic centimeter and the

earth's magnetic field $H_{DC} = .4$ gauss. For the sake of comparison the power patterns corresponding to $H_{DC} = 0$ are also shown in the same graphs.

The examination of the radiation patterns in Figures 9.7 and 9.8 reveals that the transmission of energy from the source through the plasma and into the free space is generally enhanced by the presence of the steady magnetic field especially at wave frequencies below the plasma frequency. At frequencies well above the plasma frequency, the action of the steady magnetic field becomes insignificant which is well understood. Another interesting phenomenon is the appearance of a peak in the radiation patterns at angles corresponding to the critical angles (see Fig. 9.6). As we showed in equation (9.7) these critical angles θ_{B1} and θ_{B2} are related to the indices of refraction of the ordinary and the extraordinary plane waves in magnetoplasma when the direction of propagation is along the steady magnetic field. Since the critical angles are generally different (except at frequencies high enough) for a plasma with and without the steady magnetic field (see Fig. 9.6) the corresponding patterns peak at different points.

Finally, we may mention that at certain frequencies two peaks in the radiation pattern can be expected. In our example this would occur, for instance, at a frequency of 2×10^6 cycles per second where both critical angles θ_{B1} and θ_{B2} are real (Fig. 9.6). We were not able to show this phenomenon due to the unfortunate choice of wave frequencies.

9.6 THE VALIDITY OF THE HIGH FREQUENCY APPROXIMATION

In solving the dipole problems we found it necessary to use the high frequency approximation before carrying out the integration. Since no rigorous solution to the dipole problems is available it was somewhat difficult to appraise the validity of such an approximation after the integration was

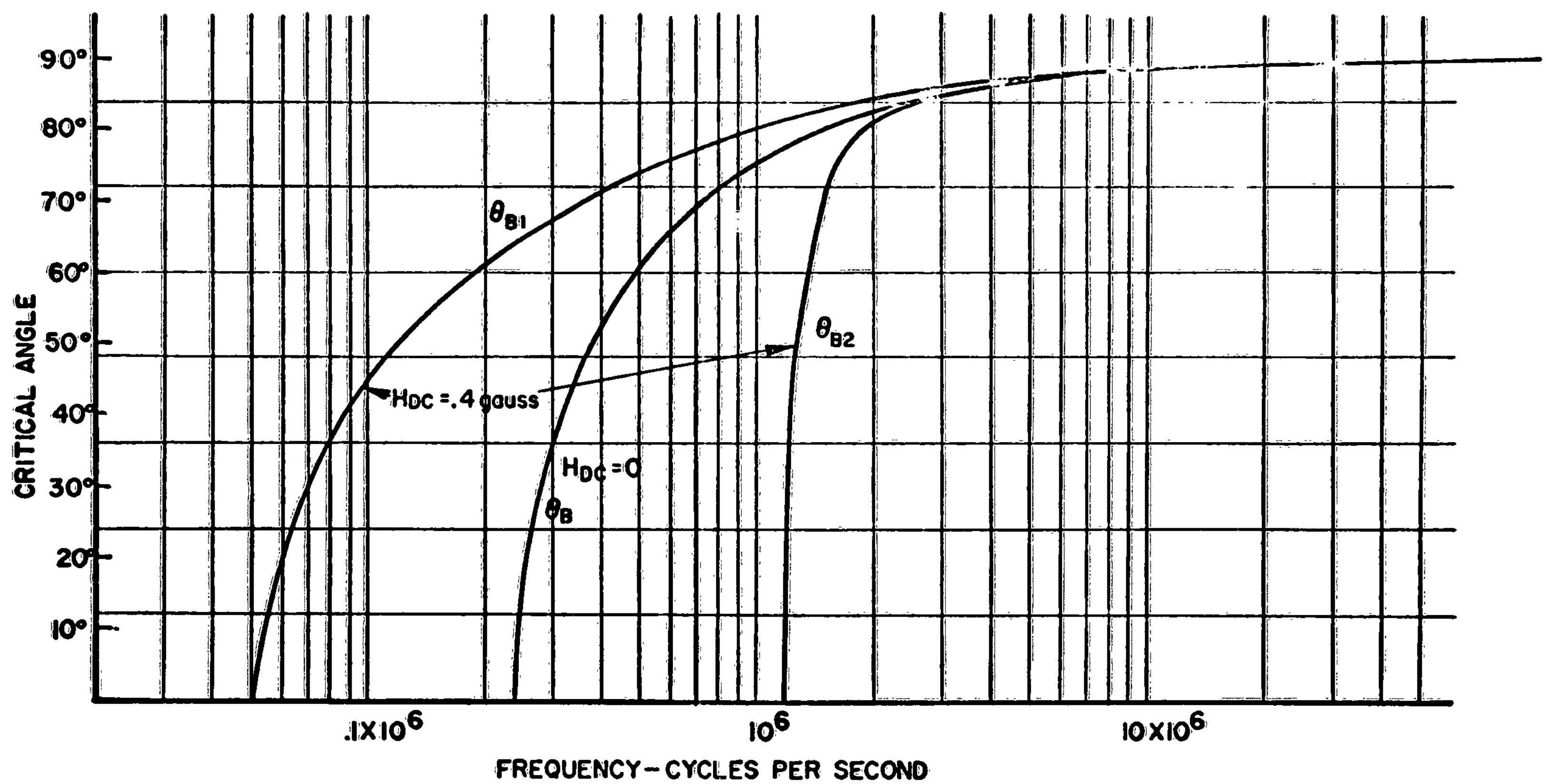

Figure 9.6 - Critical Angles Pertinent to the Problem of the Electric Current Line Source When H_{DC} is Perpendicular

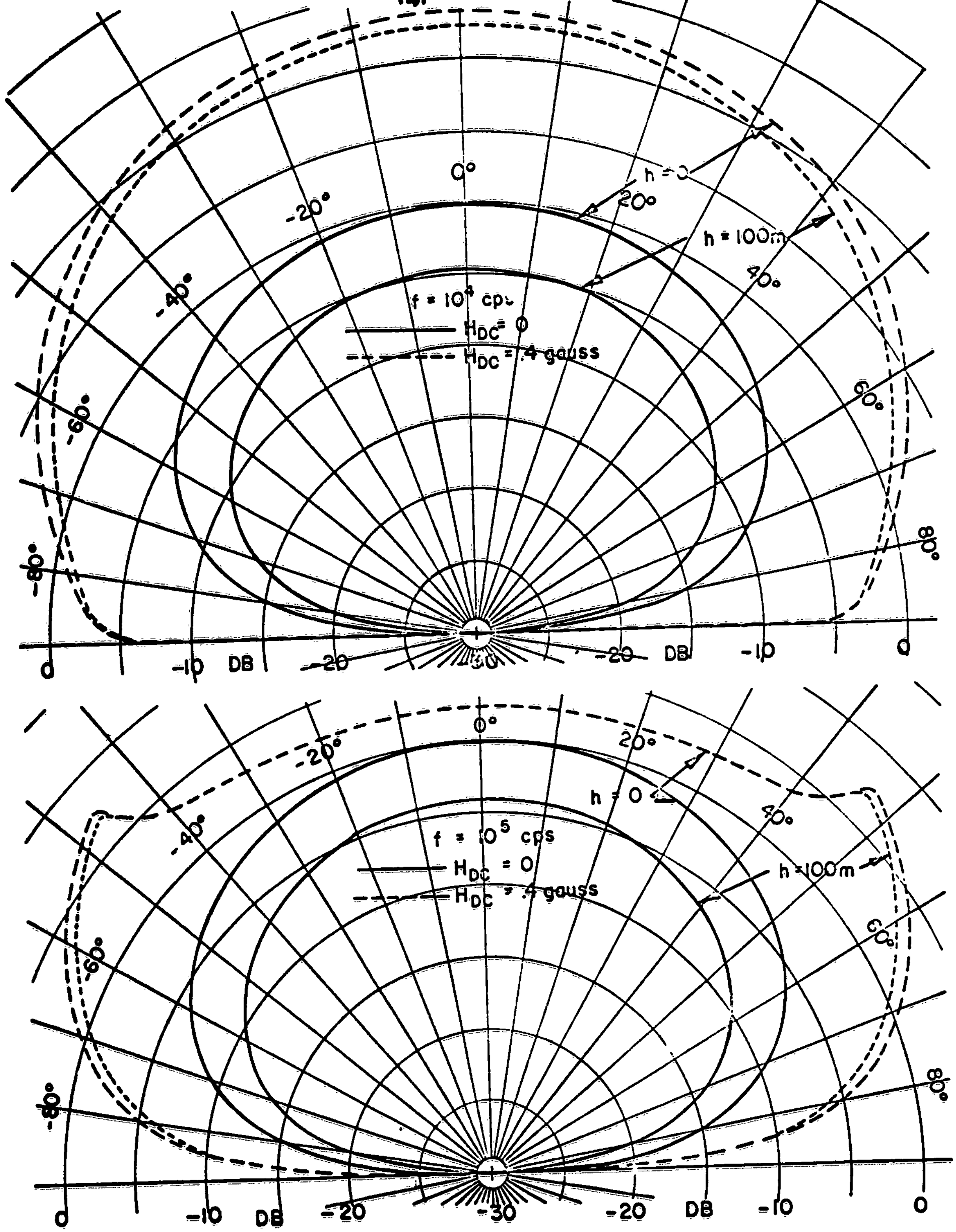

Figure 9.7 - Power Pattern in Air of an Electric Current Line Source in Magnetoplasma; H_{DC} Normal to the Line Source, N = 750 electrons per cubic centimeter, h = 0, 100 meters, H_{DC} = 0, .4 gauss, f = 10^4, 10^5 cycles per second

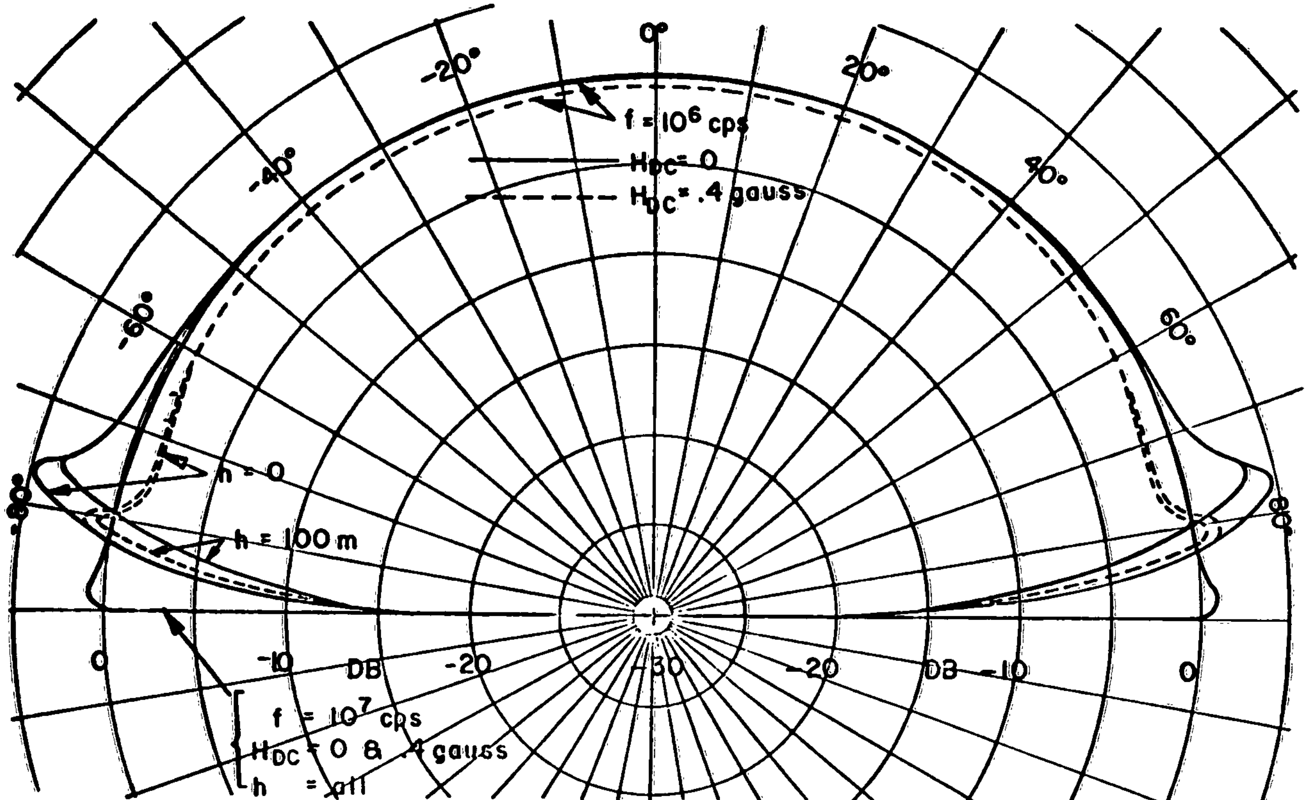

Figure 9.8 - Power Pattern in Air of an Electric Current Line Source in Magnetoplasma, H_{DC} Normal to the Line Source, N = 750 electrons per cubic centimeter, h = all, H_{DC} = 0, .4 gauss, f = 10^6, 10^7 cycles per second

carried out.

In the present case, however, it was not necessary to use any approximations to arrive at the final answer. Thus, our present solution is rigorous and it can be effectively used to test the validity of the high frequency approximation.

To this end we shall perform the following tests. First, we apply the high frequency approximation before the integration was carried out as was done in the dipole problems. Second, we apply the high frequency to the rigorous solution found in this chapter. Finally, comparing the results of the first method with that of the second, we establish the validity of this approximation.

9.6 a Introduction of the approximation before integration—The high frequency approximation of equation (3.7) in the notation of the present problem amounts to setting

$$\sigma_{1,2} = \sqrt{n^2-\sin^2\beta} \quad \pm \quad \frac{\kappa n \sin\beta}{2\sqrt{n^2-\sin^2\beta}} \tag{9.72}$$

Introducing this into the expression for the denominator $\mathcal{N}_E$ in equation (9.26) and neglecting the second order terms in κ we obtain

$$\mathcal{N}_E = -2\kappa n^3 \sin\beta \left(\sqrt{n^2-\sin^2\beta} + \cos\beta\right)\left(\sqrt{n^2-\sin^2\beta} + n^2\cos\beta\right) \tag{9.73}$$

Furthermore, it can be shown that the integrands in (9.1a), (9.1b), and (9.1c) reduce to the following:

$$E_{yo} = -\frac{\omega\mu_o I}{2\pi}\int_{\Gamma} \frac{\cos\beta\, e^{ik_o h\sqrt{n^2-\sin^2\beta}}}{\sqrt{n^2-\sin^2\beta} + \cos\beta}\, e^{ik_o\rho\cos(\beta-\theta)}\,d\beta \tag{9.74a}$$

$$E_{\theta o} = \frac{\kappa\omega\mu_o I}{4\pi k_o}\partial_x\int_{\Gamma_1}\frac{\cos\beta\cos(\beta-\theta)[1-ik_oh(\sqrt{n^2-\sin^2\beta}+\cos\beta)]e^{ik_oh\sqrt{n^2-\sin^2\beta}}}{\sqrt{n^2-\sin^2\beta}(\sqrt{n^2-\sin^2\beta}+\cos\beta)(\sqrt{n^2-\sin^2\beta}+n^2\cos\beta)}$$

$$\cdot e^{ik_o\rho\cos(\beta-\theta)}\, d\beta \tag{9.74b}$$

$$E_{\rho o} = \frac{-\kappa\omega\mu_o I}{4\pi k_o}\partial_x\int_{\Gamma_1}\frac{\cos\beta\sin(\beta-\theta)[1-ik_oh(\sqrt{n^2-\sin^2\beta}+\cos\beta)]e^{ik_oh\sqrt{n^2-\sin^2\beta}}}{\sqrt{n^2-\sin^2\beta}(\sqrt{n^2-\sin^2\beta}+\cos\beta)(\sqrt{n^2-\sin^2\beta}+n^2\cos\beta)}$$

$$\cdot e^{ik_o\rho\cos(\beta-\theta)}\, d\beta\ . \tag{9.74c}$$

The saddle-point integration can be carried out at once. Using (9.12) we obtain

$$E_{yo}^{(s)} = -\frac{\omega\mu_o I}{\sqrt{2\pi}}\cdot\frac{\cos\theta\, e^{ik_oh\sqrt{n^2-\sin^2\theta}}}{\sqrt{n^2-\sin^2\theta}+\cos\theta}\cdot\frac{e^{i(k_o\rho-\pi/4)}}{\sqrt{k_o\rho}} \tag{9.75a}$$

$$E_{\theta o}^{(s)} = \frac{i\kappa\omega\mu_o I}{4\sqrt{2\pi}}\left\{\frac{\sin 2\theta[1-ik_oh(\sqrt{n^2-\sin^2\theta}+\cos\theta)]e^{ik_oh\sqrt{n^2-\sin^2\theta}}}{\sqrt{n^2-\sin^2\theta}(\sqrt{n^2-\sin^2\theta}+\cos\theta)(\sqrt{n^2-\sin^2\theta}+n^2\cos\theta)}\right\}\frac{e^{i(k_o\rho-\pi/4)}}{\sqrt{k_o\rho}} \tag{9.75b}$$

$$E_{\rho o}^{(s)} = 0\ . \tag{9.75c}$$

We note that the integrands in (9.74a), (9.74b), and (9.74c) now contain a branch point at $\sin\theta_s = n$. To perform the branch-cut integration we use the formulation of Section 9.2b and obtain

$$E_{yo}^{(s)} = \frac{i\,\omega\mu_o I\sqrt{n}\,(1-ik_oh\sqrt{1-n^2})\,e^{-i\pi/4}\ e^{ik_o\rho\cos(|\theta|-\theta_s)}}{\sqrt{2\pi}\ \sqrt[4]{1-n^2}\ [\,k_o\rho\sin(\,|\theta|-\theta_s\,)]^{3/2}} \tag{9.76a}$$

$$E_{\theta o}^{(s)} = \frac{\mp\kappa\omega\mu_o I\sqrt{n}\,(1-ik_oh\sqrt{1-n^2})\ e^{i\pi/4}\cos(|\theta|-\theta_s)e^{ik_o\rho\cos(|\theta|-\theta_s)}}{2\ \sqrt{2\pi}\,(1-n^2)^{3/4}\ \sqrt{k_o\rho\sin(|\theta|-\theta_s)}} \tag{9.76b}$$

$$E_{\varphi o}^{(s)} = \frac{-\kappa\omega\mu_o I\sqrt{n}\,(1-ik_o h\sqrt{1-n^2})e^{i\pi/4}\sin(|\theta|-\theta_B)e^{ik_o\rho\cos(|\theta|-\theta_B)}}{2\sqrt{2\pi}\,(1-n^2)^{3/4}\sqrt{k_o\rho\sin(|\theta|-\theta_B)}} \cdot (9.76c)$$

9.6 b Introduction of the approximation after integration—Introduction of the high frequency approximation in the form

$$\sigma_{1,2} \sim \sqrt{n^2-\sin^2\theta} \pm \frac{\kappa n \sin\theta}{2\sqrt{n^2-\sin^2\theta}} \tag{9.77}$$

into the field expressions of (9.22) and (9.24) gives results that are identical with those of (9.75a), (9.75b), and (9.75c).

The application of the same approximation to the components of the lateral field in (9.35), (9.44), (9.51), (9.56), (9.59) and (9.60) affects the amplitudes of these components only but does not alter the order. Thus, all of these components remain to be of the second order.

9.6 c The region of validity of the high frequency approximation—In the preceding sections we applied the high frequency approximation before and after the integration and obtained the corresponding results. We found that in the case of the radiation field, which is given by the saddle-point contribution, the results did not differ. However, in the case of the lateral waves given by the branch-cut integration, we found that the results differed substantially.

The reason for this apparent discrepancy is as follows. The high frequency approximation introduced prior to the integration was of the form

$$\sigma_{1,2} = \sqrt{n^2-\sin^2\beta} \pm \frac{\kappa n \sin\beta}{2\sqrt{n^2-\sin^2\beta}} . \tag{9.78}$$

Obviously this approximation is not valid when the denominator of the second term vanishes, i.e., when $\sin\beta = \pm n$. Now in the saddle-point method of

integration, the process of integration merely changes β to the polar angle θ leaving everything else unchanged. Thus, the result is valid everywhere except where $\sin\theta = \pm n$ and it is independent of whether we apply this approximation before or after the integration.

In the case of branch-cut integration, however, the situation is quite different. Whereas, in the rigorous form of the integrands contain two branch points at $\sigma_1 = 0$ and $\sigma_2 = 0$, the approximation (9.78) in effect coalesces these two branch points into one, the branch point of plasma in absence of the steady magnetic field. The branch-cut integration in this case is then not valid and its result must be disregarded.

9.7 CLOSURE

In this chapter we found the field components in the air-region due to an electric current line source in a magnetoplasma when the steady magnetic field is normal to the line source. In particular, we found that the field in the air-region consists of two main components, the radiation field and the lateral field.

The radiation field contains, in addition to the components present in the corresponding isotropic case, additional transverse components of the electric and magnetic fields which render the resulting fields to be eliptically polarized everywhere except right above the source.

The lateral field is of second order as in the corresponding isotropic case, however, unlike in that case it may consist of two waves rather than one. This is due to the fact that the magnetoplasma possesses two distinct indices of refraction, corresponding to the ordinary and extraordinary waves, each one of these waves possessing its own critical angle which is the angle of the total internal reflection.

Finally, by introducing the high frequency approximation in the rigorous

expressions before and after the integration and comparing the corresponding results, we were able to determine the region of validity of this approximation. It appears that the final expression for the radiation field is not dependent on whether this approximation is introduced before or after the integration. Thus, the process of its introduction prior to the integration is valid as far as the saddle-point method of integration is concerned. In the case of the branch-cut integration, however, it is not so. Here the corresponding results differ significantly and therefore the branch-cut integration of the approximate integrand is not valid.

CHAPTER 10

THE STEADY MAGNETIC FIELD PARALLEL TO THE LINE SOURCE

In this chapter we shall be concerned with finding the electric and magnetic fields due to an electric current line source located in a magnetoplasma when the steady magnetic field is in the direction of the line source.

10.1 FORMULATION OF THE PROBLEM

The geometry of the problem is shown in Fig. 10.1. The horizontal plane $z = 0$ coincides with the interface between the anisotropic homogeneous plasma and air. For convenience we shall call the plasma medium (1) and the air medium (0). As before, we assume that both media have the same magnetic inductive capacity of free space, μ_0 . The steady magnetic field will, in this case, be oriented along the line source whereas the line source as well as the steady magnetic field will be parallel to the interface.

<u>10.1 a Fundamental equations</u>—The solution of the present boundary value problem entails solution to the Maxwell's equations in the following form:

$$\vec{\nabla} \times \vec{E}_1 = i\omega\mu_0\vec{H}_1 \tag{10.1a}$$

$$\vec{\nabla} \times \vec{H}_1 = -i\omega\epsilon_0 \overleftrightarrow{\epsilon}\, \vec{E} + I\delta(y)\,\delta(z+h)\,\vec{i}_x \tag{10.1b}$$

$$\vec{\nabla}\cdot\vec{D}_1 = \rho \tag{10.1c}$$

$$\vec{\nabla}\cdot\vec{B}_1 = 0 \tag{10.1d}$$

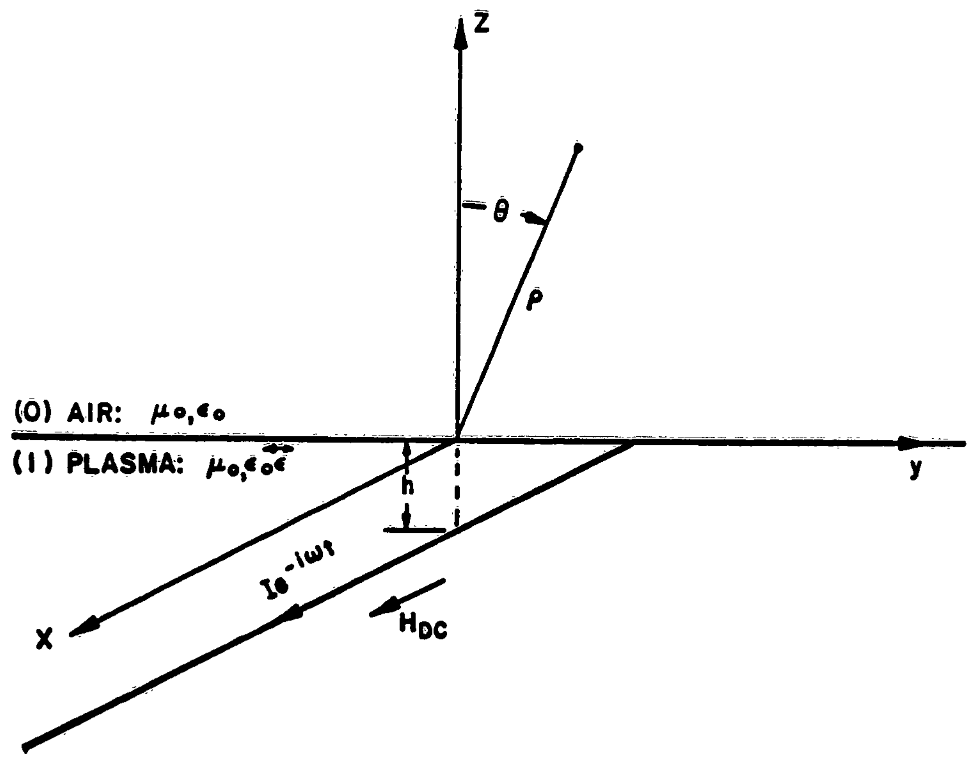

Figure 10.1 - Geometry of the Problem of the Electric Current Line Source When the Steady Magnetic Field is Parallel

in the plasma-region and

$$\vec{\nabla} \times \vec{E}_o = i\omega\mu_o \vec{H}_o$$
$$\vec{\nabla} \times \vec{H}_o = -i\omega\epsilon_o \vec{E}_o \qquad (10.2)$$
$$\vec{\nabla}\cdot\vec{D}_o = 0$$
$$\vec{\nabla}\cdot\vec{B}_o = 0$$

in the air-region. We concentrate on the plasma-region first. To obtain the wave equation we perform a curl operation on (10.1a) and subsequently using (10.1b) we obtain

$$-\vec{\nabla} \times \vec{\nabla} \times \vec{E}_1 + k_o^2 \overleftrightarrow{\epsilon}\, \vec{E}_1 = -i\omega\mu_o I\,\delta(y)\,\delta(z+h)\,\vec{1}_x . \qquad (10.3)$$

10.1 b <u>The integral representation of the field components</u>--Using the components of the tensor $\overleftrightarrow{\epsilon}$ as in equation (2.8) and carrying out the necessary algebraic operations, noting that $\partial_x = 0$, one obtains

$$\begin{bmatrix} \zeta k_o^2 + \partial_y^2 + \partial_z^2 & 0 & 0 \\ 0 & \epsilon k_o^2 + \partial_z^2 & i\eta k_o^2 - \partial_y\partial_z \\ 0 & -i\eta k_o^2 - \partial_y\partial_z & \epsilon k_o^2 + \partial_y^2 \end{bmatrix} \begin{bmatrix} E_{x1} \\ E_{y1} \\ E_{z1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot (-i\omega\mu_o I\,\delta(y)\delta(z+h)) . \qquad (10.4)$$

From (10.4) it appears that a single component of the electric field may be sufficient to solve the present boundary value problem. We write at once

$$(\zeta k_o^2 + \partial_y^2 + \partial_z^2)\, E_{x1} = -i\omega\mu_o I\,\delta(y)\,\delta(z+h) . \qquad (10.5)$$

For convenience we introduce the Fourier transform pair

$$\tilde{E}_{x1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_{x1}\, e^{-i\alpha_2 y}\, dy \qquad (10.6a)$$

$$E_{x1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{E}_{x1}\, e^{i\alpha_2 y}\, d\alpha_2 \qquad (10.6b)$$

whereupon we transform equation (10.5) as follows:

$$(\partial_z^2 + s^2)\,\tilde{E}_{x1} = -\frac{i\omega\mu_0 I}{\sqrt{2\pi}}\,\delta(z+h) \tag{10.7}$$

since the vanishing of the integrated part is assured when the radiation condition is satisfied. The parameter s is given by

$$s = \sqrt{\zeta k_0^2 - \alpha_2^2}$$
$$\operatorname{Im}\{s\} \geq 0 . \tag{10.8}$$

The solution to the differential equation (10.7) is well-known and we write it at once

$$\tilde{E}_{x1} = -\frac{\omega\mu_0 I}{2\sqrt{2\pi}}\;\frac{e^{is|z+h|}}{s} \tag{10.9}$$

Inverting with respect to the α_2 transform variable we obtain the desired result

$$E_{x1}^{(P)} = -\frac{\omega\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{e^{i(\alpha_2 y + s|z+h|)}}{s}\,d\alpha_2 \tag{10.10}$$

The form of the complementary field in the plasma immediately suggests itself, we write

$$E_{x1}^{(c)} = -\frac{\omega\mu_0 I}{4\pi}\int_{-\infty}^{\infty} A_1\, e^{i(\alpha_2 y - sz)}\,d\alpha_2 \tag{10.11}$$

and similarly for the air-region

$$E_{x0} = -\frac{\omega\mu_0 I}{4\pi}\int_{-\infty}^{\infty} A_0\, e^{i(\alpha_2 y - s_0 z)}\,d\alpha_2 \tag{10.12}$$

where

$$s_0 = \sqrt{k_0^2 - \alpha_2^2}$$
$$\operatorname{Im}\{s_0\} \geq 0 . \tag{10.13}$$

The boundary conditions to be satisfied at the interface $z = 0$ require the continuity of the tangential components of the electric and magnetic fields

This amounts to

$$E_{x0} = E_{x1}$$
$$\partial_z E_{x0} = \partial_z E_{x1} \tag{10.14}$$

which determines the coefficients A_0 and A_1 as follows:

$$A_0 = \frac{2\, e^{ish}}{s_0 + s}$$
$$A_1 = \frac{s - s_0}{s + s_0} \cdot \frac{e^{ish}}{s} \quad . \tag{10.15}$$

Thus, integral forms of the electric field components are

$$E_{x1} = -\frac{\omega\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \left\{ \frac{e^{i(\alpha_2 y + s|z+h|)}}{s} - \frac{e^{i[\alpha_2 y - s(z-h)]}}{s} + \frac{2e^{i[\alpha_2 y - s(z-h)]}}{s + s_0} \right\} d\alpha_2 \tag{10.16}$$

$$E_{x0} = -\frac{\omega\mu_0 I}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ish}}{s_0 + s}\, e^{i(\alpha_2 y + s_0 z)}\, d\alpha_2 \quad . \tag{10.17}$$

10.2 THE FIELD COMPONENTS

Having found the integral representations of the field components in each region, we shall now reduce them to the form suitable for numerical computations. In this process we shall avail ourselves of the techniques of saddle-point and branch-cut integration developed in the previous chapters.

<u>10.2 a Fields in the plasma</u>—We transform the integral expression (10.16) to cylindrical coordinates in both transform and configuration spaces using

$$\begin{aligned} \alpha_2 &= k_1 \sin\beta \\ y &= \rho_{1,2} \sin\theta_{1,2} \\ z + h &= \rho_1 \cos\theta_1 \\ z - h &= \rho_2 \cos\theta_2 \end{aligned} \tag{10.18}$$

and obtain

$$E_{x_1} = -\frac{\omega\mu_0 I}{4\pi}\int_{\Gamma}\Big\{e^{ik_1\rho_1\cos(\beta-\theta_1)} - e^{ik_1\rho_2\cos(\beta-\theta_2)} + \frac{2\cos\beta\, e^{ik_1\rho_2\cos(\beta-\theta_2)}}{\cos\beta+\sqrt{1/n^2-\sin^2\beta}}\Big\}\,d\beta \tag{10.19}$$

The first two terms in the above integral can be recognized immediately as integral representations of Hankel functions. Thus, we can rewrite (10.19) in the form

$$E_{x_1} = \frac{\omega\mu_0 I}{4}\Big\{H_0^{(1)}(k_1\rho_2) - H_0^{(1)}(k_1\rho_1)\Big\} - E_{x_1}^{(r)} \tag{10.20}$$

where

$$E_{x_1}^{(r)} = \frac{\omega\mu_0 I}{2\pi}\int_{\Gamma}\frac{\cos\beta\, e^{ik_1\rho_2\cos(\beta-\theta_2)}}{\cos\beta+\sqrt{1/n^2-\sin^2\beta}}\,d\beta\ . \tag{10.21}$$

The integrand in (10.21) has no poles in the complex β-plane but it has a branch point where the radical $(1/n^2-\sin^2\beta)^{\frac{1}{2}}$ vanishes. That is where

$$\sin\beta = \pm\frac{1}{n}\ . \tag{10.22}$$

For $|n|>1$ this branch point will lie within the region of interest and in deforming the path of integration to the path of the steepest descent through the saddle point, we must add to the saddle-point contribution the contribution from integration along the borders of this branch cut. Following the procedure of the previous chapters we write

$$E_{x_1}^{(r)} = E_{x_1}^{(rs)} - E_{x_1}^{(rb)}\, u(\theta-\theta_s) \tag{10.23}$$

where

$$\theta_s = \arcsin\left(\frac{1}{n}\right) \tag{10.24}$$

and $E_{x_1}^{(rs)}$ stand for the saddle-point contribution and $E_{x_1}^{(rb)}$ stand for the branch-cut contribution to the reflected field. Using (9.12) we obtain for the leading term of the saddle-point contribution

$$E_{x_1}^{(rs)} \sim \frac{\omega\mu_0 I}{\sqrt{2\pi}} \cdot \frac{\cos\theta}{(\cos\theta + \sqrt{\frac{1}{n^2} - \sin^2\theta})} \cdot \frac{e^{i(k_1\rho_2 - \pi/4)}}{\sqrt{k_1\rho_2}} \qquad (10.25)$$

To evaluate the branch-cut contribution we use the formulation (9.21a) and write

$$E_{x_1}^{(rB)} = ie^{ik_1\rho_2\cos(|\theta|-\theta_B)} \int_0^\infty x\, R(\pm\beta)\, e^{-k_1\rho_2\sin(|\theta|-\theta_B)x^2/2}\, dx \qquad (10.26)$$

where

$$R\,(\pm\beta\,) = F_-\,(\pm\beta\,) - F_+\,(\pm\beta\,)$$
$$\beta = \theta_B + \frac{i\,x^2}{2}\,. \qquad (10.27)$$

Now we find

$$R\,(\pm\beta\,) = \frac{\omega\mu_0 I\,\cos\beta\,\sqrt{1/n^2 - \sin^2\beta}}{\pi\left[\cos^2\beta\; - (1/n^2 - \sin^2\beta\,)\right]} \qquad (10.28)$$

We can put $\beta = \theta_B$ everywhere except in the radical which vanishes at that point. We compute it then to the next higher order of approximation and obtain

$$\sqrt{1/n^2 - \sin^2\beta} \sim -\,e^{-i\pi/4}\sqrt{1/n\,\sqrt{1-1/n^2}}\,x \qquad (10.29)$$

where of the two possible signs of the radical we have chosen the one that guarantees $\mathrm{Im}\left\{\,(1/n^2 - \sin^2\beta\,)^{\frac{1}{2}}\right\} > 0$. We find for the leading term

$$R\,(\pm\beta\,) \sim \frac{-\,\omega\mu_0 I e^{-i\pi/4}}{\pi\sqrt[4]{n^2 - 1}}\;x \qquad (10.30)$$

Substituting the above result into (10.26) and performing the integration where we use (4.31b) we obtain finally

$$E_{x_1}^{(rB)} \sim \frac{-\,\omega\mu_0 I e^{i[k_1\rho_2\cos(|\theta|-\theta_B) + \pi/4]}}{\sqrt{2\pi}\,\left[\,k_1\,\rho_2\,\sin\,(|\theta| - \theta_B\,)\right]^{3/2}} \qquad (10.31)$$

and note that the lateral field is of second order.

To summarise the above results we write the first order electric field in the plasma

$$E_{x_1} = -\frac{\omega\mu_0 I e^{-i\pi/4}}{2\sqrt{2\pi k_1}}\left\{\frac{e^{ik_1\rho_1}}{\sqrt{\rho_1}} - \frac{e^{ik_1\rho_2}}{\sqrt{\rho_2}} + \frac{2\cos\theta\, e^{ik_1\rho_2}}{\cos\theta + \sqrt{1/n^2 - \sin^2\theta}}\right\}. \tag{10.32}$$

The magnetic field in the plasma has only one component which is given by

$$H_{\theta_1} = -\frac{n}{Z_0} E_{x_1} \tag{10.33}$$

where Z_0 is the free space impedance.

<u>10.2 b Fields in the air</u>—We transform the integral expression (10.17) to cylindrical coordinates in both the transform and configuration spaces using

$$\begin{aligned} \alpha_z &= k_0 \sin\beta \\ y &= \rho \sin\theta \\ z &= \rho\cos\theta \end{aligned} \tag{10.34}$$

and obtain

$$E_{x0} = \frac{-\omega\mu_0 I}{2\pi}\int_{\Gamma_1} \frac{\cos\beta\; e^{ik_0 h\sqrt{n^2-\sin^2\beta}}}{\cos\beta + \sqrt{n^2-\sin^2\beta}}\, d\beta \tag{10.35}$$

where Γ_1 is the appropriate contour in the β-plane.

The integrand contains the radical $(n^2 - \sin^2\beta)^{\frac{1}{2}}$ as a result of which the point $\beta = \theta_s$ where

$$\theta_s = \pm \arcsin n \tag{10.36}$$

will be the branch point. Following the method of the previous sections we then write

$$E_{x0} = E_{x0}^{(s)} + E_{x0}^{(B)}\, u(\theta - \theta_s) \tag{10.37}$$

where $E_{xo}^{(s)}$ and $E_{xo}^{(B)}$ are the saddle-point and the branch-cut contributions respectively. The saddle-point contribution can be found immediately using (9.12). We obtain

$$E_{xo}^{(s)} = - \frac{\omega \mu_o I}{\sqrt{2\pi}} \cdot \frac{\cos \theta \, e^{ik_o h \sqrt{n^2 - \sin^2\theta}}}{\cos \theta + \sqrt{n^2 - \sin^2 \theta}} \quad \frac{e^{i(k_o \rho - \pi/4)}}{\sqrt{k_o \rho}} \; . \tag{10.38}$$

To evaluate the branch-cut contribution we use the formulation of (9.21a) and write

$$E_{xo}^{(B)} = i e^{ik_o \rho \cos(|\theta| - \theta_B)} \int_0^{\infty} x \, R(\pm\beta) \, e^{-k_o \rho \sin(|\theta| - \theta_B) x^2/2} \, dx \tag{10.39}$$

where

$$R\,(\pm\beta) = F_-\,(\pm\beta) - F_+\,(\pm\beta) \tag{10.40}$$

and

$$\beta = \theta_B + \frac{i x^2}{2}$$

Now we find

$$R(\pm\beta) = \frac{-\omega\mu_o I}{\pi} \cos\beta \left\{ \frac{\sqrt{n^2 - \sin^2\beta} \;\; \cos k_o h \sqrt{n^2 - \sin^2\beta} \; - i \sin k_o h \sqrt{n^2 - \sin^2\beta}}{\cos^2\beta - (n^2 - \sin^2\beta)} \right\} \tag{10.41}$$

We set $\beta = \theta_B$ everywhere except in the radical which becomes zero at that point. We compute it then to the next higher approximation and obtain

$$\sqrt{n^2 - \sin^2\beta} \sim - e^{-i\pi/4} \cdot \sqrt{n \sqrt{1 - n^2}} \, x \tag{10.42}$$

where of the two possible choices of the sign for the radical we have chosen the one that makes $\mathrm{Im} \left\{ (n^2 - \sin^2\beta)^{\frac{1}{2}} \right\} > 0$. Substituting the last expression into (10.41) we obtain for the leading term

$$R\,(\pm\beta) \sim \frac{\omega\mu_o I}{\pi} \sqrt[4]{\frac{n^2}{1 - n^2}} \, (1 - ik_o h) \, e^{-i\pi/4} \; x \; . \tag{10.43}$$

Using this result in the integrand of (10.39) gives finally

$$E^{(s)}_{xo} = \frac{\omega\mu_o I}{\sqrt{2\pi}} \cdot \sqrt[4]{\frac{n^2}{1-n^2}} \cdot (1 - ik_o h) \frac{e^{i[k_o\rho\cos(|\theta|-\theta_s)+\pi/4]}}{[k_o\rho \sin(|\theta| - \theta_s)]^{3/2}}. \qquad (10.44)$$

and note that this lateral field is of second order.

The magnetic field components can be found from the appropriate curl equation. The only non-vanishing component is $H_{\theta o}$ which is given by

$$H_{\theta o} = -\frac{E_{xo}}{Z_o} \qquad (10.45)$$

where Z_o is the free-space impedance.

The Poynting vector has a single component given by

$$S_\rho = \frac{\omega\mu_o I^2}{4\pi\rho} \left| \frac{\cos\theta\, e^{ik_o h\sqrt{n^2-\sin^2\theta}}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}} \right|^2 . \qquad (10.46)$$

10.3 CLOSURE

In this chapter we have formulated and solved the problem of an electric current line source situated in a magnetoplasma when the steady magnetic field is along the line source and the line source is parallel to the boundary. In particular we found the radiation in the air and in the magnetoplasma as well as the lateral field in each region when conditions are favorable for its existence. In this problem the steady magnetic field has no effect on the plasma since the electric field vector is always in the same direction as the steady magnetic field.

PART III

FIELD OF MAGNETIC CURRENT LINE SOURCES IN MAGNETOPLASMA WITH A SEPARATION BOUNDARY

CHAPTER 11

RIGOROUS FORMULATION OF THE PROBLEM OF A MAGNETIC CURRENT LINE SOURCE WHEN THE STEADY MAGNETIC FIELD IS NORMAL TO IT

In this chapter we shall be concerned with the finding of appropriate integral representations for the Cartesian components of the field vectors for the magnetoplasma- and air-half-spaces. This will entail a review of the fundamental field equations and the definition of the source of the electromagnetic waves.

11.1 STATEMENT OF THE PROBLEM

The geometry of the problem is shown in Figure 11.1. The horizontal plane $z = 0$ coincides with the interface between the anisotropic homogeneous plasma and air. For convenience we call the plasma medium (1) and the air medium (0). As before, we assume that both media have the same magnetic inductive capacity of free space, μ_o. The steady magnetic field H_{DC} will, in this case, be oriented normal to the line source whereas the plane formed by the line source and the direction of the steady magnetic field will be parallel to the interface.

11.2 FUNDAMENTAL EQUATIONS

The definition of the present boundary valve problem implies the solution to Maxwell's equations subject to the usual boundary conditions at the interface and proper behavior at infinity. The use of the auxiliary vector

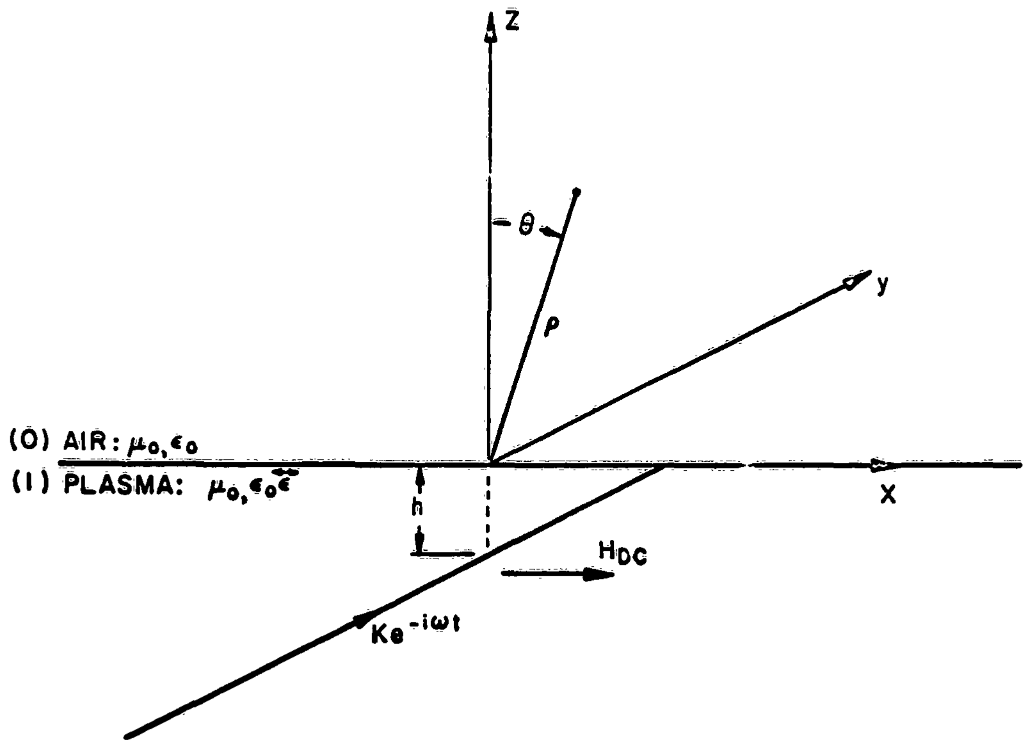

Figure 11.1 - Geometry of the Problem of a Magnet..c Current Line Source When The Steady Magnetic Field is Perpendicular

potentials does not seem to simplify the problem any. Thus, we shall work with the field components directly.

<u>11.2 a The nature of the source</u>—For the purpose of this problem it will be assumed that the source of the electromagnetic waves consists of a very thin straight solenoid of infinite extent carrying an alternating current $Ie^{-i\omega t}$. It is well-known (25,p.185) that such a solenoid can be replaced for convenience by a line carrying a magnetic current $Ke^{-i\omega t}$. To localize the source properly we shall write for the magnetic current density

$$\vec{J}_m = K\,\delta(x)\,\delta(z+h)\,\vec{1}_y \tag{11.1}$$

where $\delta(\;)$ is the Dirac delta function.

<u>11.2 b The field equations</u>—As we remarked before, our source of electromagnetic waves in this problem may be regarded as a magnetic current line singularity at a point $x = 0$, $z = -h$. Then for the magnetoplasma region, the appropriate form of the Maxwell's equations is

$$\begin{aligned} \vec{\nabla} \times \vec{E}_1 &= i\,\omega\mu_0 \vec{H}_1 - K\,\delta(x)\,\delta(z+h)\,\vec{1}_y \\ \vec{\nabla} \times \vec{H}_1 &= -i\,\omega\epsilon_0 \overleftrightarrow{\epsilon}\,\vec{E}_1 \\ \vec{\nabla}\cdot\vec{D}_1 &= 0 \\ \vec{\nabla}\cdot\vec{B}_1 &= \rho_m \end{aligned} \tag{11.2}$$

where ρ_m is the fictitious "magnetic charge density" related to the magnetic current density by the continuity equation

$$\vec{\nabla}\cdot\vec{J}_m = -\partial_t \rho_m . \tag{11.3}$$

Since $\overleftrightarrow{\epsilon}$ is non-singular, the second equation of (11.2) can be written

$$\overleftrightarrow{\epsilon}^{-1}\vec{\nabla}\times\vec{H}_1 = -i\,\omega\epsilon_0\vec{E}_1 . \tag{11.4}$$

Now performing a curl operation on the last equation and using the first equation of (11.2), we obtain the wave equation

$$-\vec{\nabla}\times\vec{\epsilon}^{\,-1}\,\vec{\nabla}\times\vec{H}_1+k_o^2\,\vec{H}_1=-i\omega\epsilon_o K\,\delta(x)\,\delta(z+h)\,\vec{1}_y \tag{11.5}$$

Using the inverse of the permittivity tensor of equation (2.9) and carrying out the necessary algebraic operation where we note that $\partial_y = 0$, we obtain a set of simultaneous equations as follows:

$$\begin{bmatrix} \chi k_o^2+\partial_z^2 & -i\kappa\,\partial_x\partial_z & -\partial_x\partial_z \\ i\kappa\,\partial_x\partial_z & \chi k_o^2+\partial_x^2+\frac{\chi}{\varsigma}\partial_z^2 & -i\kappa\,\partial_x^2 \\ -\partial_x\partial_z & i\kappa\,\partial_x^2 & \chi k_o^2+\partial_x^2 \end{bmatrix}\begin{bmatrix} H_{x1} \\ H_{y1} \\ H_{z1} \end{bmatrix}=-i\omega\epsilon_o\chi K\delta(x)\delta(z+h)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{11.6}$$

which we shall leave in this form for the time being. From equation (11.4) we note that the electric field components can be expressed in terms of the magnetic field components as follows:

$$\begin{bmatrix} E_{x1} \\ E_{y1} \\ E_{z1} \end{bmatrix}=\frac{1}{i\omega\epsilon_o\chi}\begin{bmatrix} 0 & \frac{\chi}{\varsigma}\partial_z & 0 \\ -\partial_z & i\kappa\partial_x & \partial_x \\ -i\kappa\partial_z & -\partial_x & i\kappa\partial_x \end{bmatrix}\begin{bmatrix} H_{x1} \\ H_{y1} \\ H_{z1} \end{bmatrix} \tag{11.7}$$

In the air-region the appropriate form of the Maxwell's equations is

$$\begin{aligned} \vec{\nabla}\times\vec{E}_o &= i\omega\mu_o\vec{H}_o \\ \vec{\nabla}\times\vec{H}_o &= -i\omega\epsilon_o\vec{E}_o \\ \vec{\nabla}\cdot\vec{D}_o &= 0 \\ \vec{\nabla}\cdot\vec{B}_o &= 0 \end{aligned} \tag{11.8}$$

and the Cartesian components of the magnetic and electric fields satisfy the vector wave equation

$$(\nabla^2+k_o^2)\begin{Bmatrix} \vec{H}_o \\ \vec{E}_o \end{Bmatrix}=0\,. \tag{11.9}$$

11.3 FOURIER INTEGRAL REPRESENTATION IN CARTESIAN COORDINATES

The formulation of the present boundary value problem can be simplified a great deal by expressing the field components in the magnetoplasma and in the air in terms of their double Fourier integral representation in Cartesian coordinates in the transform space as well as in the configuration space. Thus, as before, we introduce a double Fourier transform pair defined by

$$\tilde{\tilde{F}}(\alpha_1, \alpha_3) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} F(x, z)\, e^{-i(\alpha_1 x + \alpha_3 z)}\, dx dz \qquad (11.10)$$

and

$$F(x, z) = \frac{1}{2\pi} \iint_{-\infty}^{\infty} \tilde{\tilde{F}}(\alpha_1, \alpha_3)\, e^{i(\alpha_1 x + \alpha_3 z)}\, d\alpha_1\, d\alpha_3 \quad . \qquad (11.11)$$

In what follows we shall also need the transforms of the derivatives. These can be obtained by integrating by parts. As was shown in Chapter 2 for the triple transform, the vanishing of the integrated part at the upper and lower limits is assured providing the radiation condition is satisfied. Thus, we can establish the following correspondences:

$$\begin{aligned} \partial_x F &\Longleftrightarrow i\, \alpha_1 \tilde{\tilde{F}} \\ \partial_z F &\Longleftrightarrow i\, \alpha_3 \tilde{\tilde{F}} . \end{aligned} \qquad (11.12)$$

<u>11.3 a The particular integral corresponding to the source</u>—To transform the inhomogeneous system of the simultaneous differential equations in (11.6), one multiplies both sides by $\exp\{ i\, (\alpha_1 x + \alpha_3 z) \}$ and integrates with respect to the real variables x and z between $-\infty$ and $+\infty$. The right-hand side of (11.6) yields at once

$$\iint_{-\infty}^{\infty} \delta(x)\, \delta(z + h)\, e^{-i(\alpha_1 x + \alpha_3 z)}\, dx dz = e^{i\alpha_3 h} \qquad (11.13)$$

and the left-hand side transforms according to (11.12). Consequently, one obtains

$$\begin{bmatrix} \chi k_0^2-\alpha_3^2 & i\kappa\alpha_1\alpha_3 & \alpha_1\alpha_3 \\ -i\kappa\alpha_1\alpha_3 & \chi k_0^2-\alpha_1^2-\frac{\chi}{\zeta}\alpha_3^2 & i\kappa\alpha_1^2 \\ \alpha_1\alpha_3 & -i\kappa\alpha_1^2 & \chi k_0^2-\alpha_1^2 \end{bmatrix} \begin{bmatrix} \tilde{\tilde{H}}_{x_1} \\ \tilde{\tilde{H}}_{y_1} \\ \tilde{\tilde{H}}_{z_1} \end{bmatrix} = \frac{-i\omega\epsilon_0\chi K}{2\pi} \begin{bmatrix} 0 \\ e^{i\alpha_3 h} \\ 0 \end{bmatrix} \tag{11.14}$$

The above system of algebraic equations can be reduced to a simpler one as follows:

$$\begin{bmatrix} \alpha_1 & 0 & \alpha_3 \\ 0 & (\alpha_3^2-s_1^2)(\alpha_3^2-s_2^2) & 0 \\ 0 & -i\kappa\alpha_1^2 & \chi k_0^2-\alpha_1^2-\alpha_3^2 \end{bmatrix} \begin{bmatrix} \tilde{\tilde{H}}_{x_1} \\ \tilde{\tilde{H}}_{y_1} \\ \tilde{\tilde{H}}_{z_1} \end{bmatrix} = \frac{-i\omega\epsilon_0\zeta K(\chi k_0^2-\alpha_1^2-\alpha_3^2)}{2\pi} \begin{bmatrix} 0 \\ e^{i\alpha_3 h} \\ 0 \end{bmatrix} \tag{11.15}$$

where s_1 and s_2 are given by (8.15). Now, we can write each transformed component of the magnetic field as follows:

$$\tilde{\tilde{H}}_{y_1}^{(p)} = -\frac{i\omega\epsilon_0\zeta K}{2\pi}\left[\frac{\chi k_0^2-\alpha_1^2-\alpha_3^2}{(\alpha_3^2-s_1^2)(\alpha_3^2-s_2^2)}\right] e^{i\alpha_3 h} \tag{11.16a}$$

$$\tilde{\tilde{H}}_{z_1}^{(p)} = \frac{\omega\epsilon_0\zeta\kappa K}{2\pi}\left[\frac{\alpha_1^2}{(\alpha_3^2-s_1^2)(\alpha_3^2-s_2^2)}\right] e^{i\alpha_3 h} \tag{11.16b}$$

$$\tilde{\tilde{H}}_{x_1}^{(p)} = -\frac{\omega\epsilon_0\zeta\kappa K}{2\pi}\left[\frac{\alpha_1\alpha_3}{(\alpha_3^2-s_1^2)(\alpha_3^2-s_2^2)}\right] e^{i\alpha_3 h}. \tag{11.16c}$$

The inversion with respect to the α_1 transform variable can be performed immediately. According to (2.53) and (2.54), we obtain

$$\tilde{H}_{y_1}^{(p)} = \frac{\omega\epsilon_0\zeta K}{2\sqrt{2\pi}}\left\{\frac{\chi k_0^2-\alpha_1^2-s_1^2}{s_1(s_1^2-s_2^2)}e^{is_1|z+h|} - \frac{\chi k_0^2-\alpha_1^2-s_2^2}{s_2(s_1^2-s_2^2)}e^{is_2|z+h|}\right\} \tag{11.17a}$$

$$\tilde{H}_{z_1}^{(p)} = \frac{i\omega\epsilon_0\zeta\kappa K}{2\sqrt{2\pi}}\,\alpha_1^2\left[\frac{e^{is_1|z+h|}}{s_1(s_1^2-s_2^2)} - \frac{e^{is_2|z+h|}}{s_2(s_1^2-s_2^2)}\right] \qquad (11.17b)$$

$$\underset{z+h\gtrless 0}{\tilde{H}_{x_1}^{(p)}} = \mp\frac{i\omega\epsilon_0\zeta\kappa K}{2\sqrt{2\pi}}\,\alpha_1\left[\frac{e^{is_1|z+h|}}{s_1^2-s_2^2} - \frac{e^{is_2|z+h|}}{s_1^2-s_2^2}\right]. \qquad (11.17c)$$

Finally, we invert with respect to the α_1 transform variable and obtain the desired representation

$$H_{y_1}^{(p)} = \frac{\omega\epsilon_0\zeta K}{4\pi}\int_{-\infty}^{\infty}\left\{\frac{\chi k_0^2-\alpha_1^2-s_1^2}{s_1(s_1^2-s_2^2)}\,e^{is_1|z+h|} - \frac{\chi k_0^2-\alpha_1^2-s_2^2}{s_2(s_1^2-s_2^2)}\,e^{is_2|z+h|}\right\} e^{i\alpha_1 x}\,d\alpha_1$$

(11.18a)

$$H_{z_1}^{(p)} = \frac{-i\omega\epsilon_0\kappa K}{4\pi}\,\partial_x^2\int_{-\infty}^{\infty}\left\{\frac{e^{is_1|z+h|}}{s_1(s_1^2-s_2^2)} - \frac{e^{is_2|z+h|}}{s_2(s_1^2-s_2^2)}\right\} e^{i\alpha_1 x}\,d\alpha_1 \qquad (11.18b)$$

$$\underset{z+h\gtrless 0}{H_{x_1}^{(p)}} = \mp\frac{\omega\epsilon_0\zeta\kappa K}{4\pi}\,\partial_x\int_{-\infty}^{\infty}\left\{\frac{e^{is_1|z+h|}}{s_1^2-s_2^2} - \frac{e^{is_2|z+h|}}{s_1^2-s_2^2}\right\} e^{i\alpha_1 x}\,d\alpha_1\,. \qquad (11.18c)$$

<u>11.3 b The complementary field in the plasma</u>—In the preceding subsection we obtain the "particular integrals" of the system of equations in (11.15) which represent the primary excitation due to the source. To satisfy the boundary conditions of the problem, we shall need an appropriate complementary solution of corresponding homogeneous system.

From (11.15) it is clear that $\vec{H}_1^{(c)}$ satisfies

$$(\alpha_3^2 - s_1^2)(\alpha_3^2 - s_2^2)\,\vec{H}^{(c)} = 0 \qquad (11.19)$$

where in the above equation α_3 is looked upon as a differential operator with

respect to the z- coordinate. The solution to (11.19) can be written immediately in component form

$$\begin{aligned}
\widetilde{H}_{x1}^{(c)} &= C_{11} \quad e^{-is_1z} + C_{12} \quad e^{-is_2z} \\
\widetilde{H}_{y1}^{(c)} &= C_{21} \quad e^{-is_1z} + C_{22} \quad e^{-is_2z} \\
\widetilde{H}_{z1}^{(c)} &= C_{31} \quad e^{-is_1z} + C_{32} \quad e^{-is_2z}
\end{aligned} \tag{11.20}$$

where we discarded solutions with positive exponentials since we can have only waves going away from the interface upon reflection. The coefficients C_{ij} can be found by a method identical to the one of Section 2.3. We obtain

$$C_{11} = \frac{i\kappa\alpha_1 s_1}{\chi k_0^2 - \alpha_1^2 - s_1^2} C_{21} \tag{11.21a}$$

$$C_{31} = \frac{i\kappa\alpha_1^2}{\chi k_0^2 - \alpha_1^2 - s_1^2} C_{21} \tag{11.21b}$$

$$C_{12} = \frac{i\kappa\alpha_1 s_2}{\chi k_0^2 - \alpha_1^2 - s_2^2} C_{22} \tag{11.21c}$$

$$C_{32} = \frac{i\kappa\alpha_1^2}{\chi k_0^2 - \alpha_1^2 - s_2^2} C_{22} \; . \tag{11.21d}$$

For convenience in what follows, we shall normalize the coefficients C_{21} and C_{22} as follows:

$$C_{21} = \frac{\omega\epsilon_0\zeta K}{2\sqrt{2\pi}} A_1 \tag{11.22a}$$

$$C_{22} = \frac{\omega\epsilon_0\zeta K}{2\sqrt{2\pi}} A_2 \; . \tag{11.22b}$$

Now we invert with respect to the α_3 transform variable and obtain the desired representation

$$H_{y1}^{(c)} = \frac{\omega\epsilon_0\zeta K}{4\pi}\int_{-\infty}^{\infty}(A_1 e^{-is_1 z} + A_2 e^{-is_2 z})\, e^{i\alpha_1 x}\, d\alpha_1 \qquad (11.23a)$$

$$H_{z1}^{(c)} = \frac{-i\omega\epsilon_0\zeta\kappa K}{4\pi}\,\partial_x^2\int_{-\infty}^{\infty}\left\{\frac{A_1 e^{-is_1 z}}{\chi k_0^2 - \alpha_1^2 - s_1^2} + \frac{A_2 e^{-is_2 z}}{\chi k_0^2 - \alpha_1^2 - s_2^2}\right\} e^{i\alpha_1 x}\, d\alpha_1 \qquad (11.23b)$$

$$H_{x1}^{(c)} = \frac{\omega\epsilon_0\zeta\kappa K}{4\pi}\,\partial_x\int_{-\infty}^{\infty}\left\{\frac{s_1 A_1 e^{-is_1 z}}{\chi k_0^2 - \alpha_1^2 - s_1^2} + \frac{s_2 A_2 e^{-is_2 z}}{\chi k_0^2 - \alpha_1^2 - s_2^2}\right\} e^{i\alpha_1 x}\, d\alpha_1\,. \qquad (11.23c)$$

11.3 c The field in the air—For the air-region we choose the following solution to the wave equation (11.9):

$$H_{y0} = \frac{\omega\epsilon_0\zeta K}{4\pi}\int_{-\infty}^{\infty} B_1\, e^{i(\alpha_1 x + s_0 z)}\, d\alpha_1 \qquad (11.24a)$$

$$H_{z0} = -\frac{i\,\omega\epsilon_0\zeta\kappa K}{4\pi}\,\partial_x^2\int_{-\infty}^{\infty}\frac{B_2}{s_0}\, e^{i(\alpha_1 x + s_0 z)}\, d\alpha_1 \qquad (11.24b)$$

$$H_{x0} = -\frac{\omega\epsilon_0\zeta\kappa K}{4\pi}\,\partial_x\int_{-\infty}^{\infty} B_2 e^{i(\alpha_1 x + s_0 z)}\, d\alpha_1 \qquad (11.24c)$$

where s_0 has the same meaning as before. One can readily show that the above field components also satisfy the divergence equation $\vec{\nabla}\cdot\vec{H} = 0$.

11.4 THE BOUNDARY CONDITIONS

In the preceding sections we found field components in the magnetoplasma and air-regions which are solutions to Maxwell's equations. Moreover, in solving Maxwell's equations we chose such solutions for the field representation that have proper behavior at infinity by requiring that the imaginary part of the pertinent exponents be non-negative. In addition, the field components contain certain thus far undetermined coefficients which, upon imposition of the boundary conditions, will be determined and thus render the solution

unique.

11.4 a Statement of the boundary conditions—The boundary conditions to be satisfied by the Cartesian components of the field vectors require continuity of the tangential components of the electric and magnetic fields at the interface $z = 0$. In terms of the magnetic field only these can be written:

$$H_{x0} = H_{x1} \tag{11.25a}$$

$$H_{y0} = H_{y1} \tag{11.25b}$$

$$\zeta\, \partial_z H_{y0} = \partial_z H_{y1} \tag{11.25c}$$

$$\chi\left(\partial_z H_{x0} - \partial_x H_{z0} \right) = \partial_z H_{x1} - i\kappa\, \partial_x H_{y1} - \partial_x H_{z1}\,. \tag{11.25d}$$

The above equations can be recast in a more convenient form as follows:

$$0 = \zeta\, s_0\, \check{H}_{y1} + i\, \partial_z \check{H}_{y1} \tag{11.26a}$$

$$0 = \frac{\chi k_0^2}{s_0} \check{H}_{x1} + i\, \partial_z \check{H}_{x1} + i\kappa\alpha_1 \check{H}_{y1} + \alpha_1 \check{H}_{z1} \tag{11.26b}$$

when $\vee$ denotes the field representations with the integral sign removed.

11.4 b Application of the boundary conditions—Performing the necessary algebraic operations to satisfy equations (11.27a) and (11.26b) gives a set of two simultaneous equations in two unknowns as follows:

$$\begin{bmatrix} s_1 + \zeta s_0 & s_2 + \zeta s_0 \\ \dfrac{s_1 + s_0}{P_H(s_1)} & \dfrac{s_2 + s_0}{P_H(s_2)} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} \dfrac{s_2(s_1 - \zeta s_0) P_H(s_1) e^{is_1h} - s_1(s_2 - \zeta s_0) P_H(s_2) e^{is_2h}}{s_1 s_2 (s_1^2 - s_2^2)} \\ \dfrac{s_2(s_1 - s_0) e^{is_1h} - s_1(s_2 - s_0) e^{is_2h}}{s_1 s_2 (s_1^2 - s_2^2)} \end{bmatrix} \tag{11.27}$$

where

$$P_H(s_{1,2}) = \chi k_0^2 - \alpha_1^2 - s_{1,2}^2 \tag{11.28}$$

We solve the above system of equations using Cramer's rule and obtain

$$A_1 = \frac{P_H(s_1)\left[(s_1-\zeta s_0)(s_2+s_0)P_H(s_1)-(s_2+\zeta s_0)(s_1-s_0)P_H(s_2)\right] e^{is_1h}}{s_1(s_1^2-s_2^2)\,N_H}$$

$$-\frac{P_H(s_1)P_H(s_2)\left[(s_2-\zeta s_0)(s_2+s_0)-(s_2+\zeta s_0)(s_2-s_0)\right] e^{is_2h}}{s_2(s_1^2-s_2^2)\,N_H} \qquad (11.29a)$$

and

$$A_2 = \frac{P_H(s_2)\left[(s_2-\zeta s_0)(s_1+s_0)P_H(s_2)-(s_1+\zeta s_0)(s_2-s_0)P_H(s_1)\right] e^{is_2h}}{s_2(s_1^2-s_2^2)\,N_H}$$

$$-\frac{P_H(s_1)P_H(s_2)\left[(s_1-\zeta s_0)(s_1+s_0)-(s_1+\zeta s_0)(s_1-s_0)\right] e^{is_1h}}{s_1(s_1^2-s_2^2)\,N_H} \qquad (11.29b)$$

where

$$N_H = (s_1+\zeta s_0)(s_2+s_0)P_H(s_1)-(s_2+\zeta s_0)(s_1+s_0)P_H(s_2). \qquad (11.30)$$

To find B_1 and B_2 we use (11.25a) and (11.25b) as well as the above results and obtain

$$B_1 = -2\cdot\frac{(s_2+s_0)P_H(s_1)e^{is_1h} - (s_1+s_0)P_H(s_2)\,e^{is_2h}}{N_H} \qquad (11.31a)$$

$$B_2 = -2\,s_0\cdot\frac{(s_2+\zeta s_0)\,e^{is_1h} - (s_1+\zeta s_0)\,e^{is_2h}}{N_H}\,. \qquad (11.31b)$$

11.5 FIELD COMPONENTS IN AIR IN CYLINDRICAL COORDINATES

Having found rigorous expressions for the field components in the Cartesian coordinates in both regions, we can now transform them to cylindrical coordinates in which these field components will have somewhat simpler form.

We transform to cylindrical coordinates in both configuration and transform spaces using

$$\alpha_1 = k_o \sin\beta$$
$$z = \rho \cos\theta \qquad (11.32)$$
$$x = \rho \sin\theta$$

As a consequence we obtain

$$H_{yo} = -\frac{\omega\epsilon_o\zeta k_o K}{2\pi}\int_{\Gamma_1}\frac{(s_2+s_o)P_H(s_1)e^{is_1h}-(s_1+s_o)P_H(s_2)e^{is_2h}}{N_H}\cos\beta\, e^{ik_o\rho\cos(\beta-\theta)}d\beta \qquad (11.33a)$$

$$H_{\theta o} = \frac{\omega\epsilon_o\zeta k_o^2\kappa K}{2\pi}\partial_x\int_{\Gamma_1}\frac{(s_2+\zeta s_o)e^{is_1h}-(s_1+\zeta s_o)e^{is_2h}}{N_H}\cos\beta\cos(\beta-\theta)e^{ik_o\rho\cos(\beta-\theta)}\,d\beta \qquad (11.33b)$$

$$H_{\varphi o} = \frac{-\omega\epsilon_o k_o^2\zeta\kappa K}{2\pi}\partial_x\int_{\Gamma_1}\frac{(s_2+\zeta s_o)e^{is_1h}-(s_1+\zeta s_o)e^{is_2h}}{N_H}\cos\beta\sin(\beta-\theta)e^{ik_o\rho\cos(\beta-\theta)}\,d\beta \qquad (11.33c)$$

where Γ_1 is the integration path in the complex β-plane as shown in Fig. 3.1. In the above equations it is understood that s_1, s_2, and s_o are all functions of β according to the transformation (11.32).

11.6 CLOSURE

In the foregoing chapter we formulated rigorously the problem of a magnetic current line source in a magnetoplasma when the steady magnetic field is normal to it. In particular, we found the integral representation for the field components in the Cartesian and cylindrical coordinates in both regions. Unlike that in the corresponding isotropic case, all of the electric and magnetic field components are present in both regions.

CHAPTER 12

RESULTS FOR THE AIR REGION WHEN THE STEADY MAGNETIC FIELD IS NORMAL TO THE LINE SOURCE

Having formulated the problem in the previous chapter, we are now ready to evaluate the definite integrals to obtain explicit forms for the electric and magnetic field components.

Since the present problem is very similar to the corresponding problem of an electric line source of Chapters 9 and 10, we shall use many of those results that are applicable to the present problem.

12.1 EVALUATION OF THE FIELD INTEGRALS

The field integrals to be evaluated have the following form:

$$H_{yo} = -\frac{\omega\epsilon_o \zeta K}{2\pi} \int_{\Gamma_1} G_{yo} \cos\beta \, e^{ik_o\rho\cos(\beta-\theta)} \, d\beta \tag{12.1a}$$

$$H_{\theta o} = \frac{\omega\epsilon_o \zeta \kappa K}{2\pi k_o} \partial_x \int_{\Gamma_1} G_{\theta o} \cos\beta \cos(\beta-\theta) e^{ik_o\rho\cos(\beta-\theta)} \, d\beta \tag{12.1b}$$

$$H_{\rho o} = -\frac{\omega\epsilon_o \zeta \kappa K}{2\pi k_o} \partial_x \int_{\Gamma_1} G_{\theta o} \cos\beta \sin(\beta-\theta) e^{ik_o\rho\cos(\beta-\theta)} \, d\beta \tag{12.1c}$$

where

$$G_{yo} = \frac{(\sigma_2 + \cos\beta)P_H(\sigma_1)e^{ik_oh\sigma_1} - (\sigma_1 + \cos\beta)P_H(\sigma_2)e^{ik_oh\sigma_2}}{\gamma\tau_H} \tag{12.2a}$$

and

$$\mathcal{N}_H = (\sigma_1 + \zeta\cos\beta)(\sigma_2 + \cos\beta)P_H(\sigma_1) - (\sigma_2 + \zeta\cos\beta)(\sigma_1 + \cos\beta)P_H(\sigma_2) \qquad (12.3a)$$

$$P_H(\sigma_{1,2}) = \chi - \sin^2\beta - \sigma_{1,2}^2 \qquad (12.3b)$$

and σ_1 and σ_2 are given by (9.2a). We also note that $P_H(\sigma_1)$ and $P_H(\sigma_2)$ can be written in the form

$$P_H(\sigma_{1,2}) = \frac{1}{\epsilon}\left\{\left[\frac{\epsilon^2-\eta^2-\epsilon\zeta-(\epsilon-\zeta)\sin^2\beta}{2}\right] \mp \sqrt{\left[\frac{\epsilon^2-\eta^2-\epsilon\zeta-(\epsilon-\zeta)\sin^2\beta}{2}\right]^2 + \eta^2\zeta\sin^2\beta}\right\} \qquad (12.4)$$

<u>12.1 a Evaluation of the field integrals at the saddle point</u>--Applying the formulation (9.12) to the field integrals in (12.1) we obtain

$$H_{y0}^{(s)} = -\frac{\omega\epsilon_0\zeta K}{\sqrt{2\pi}} G_{y0}(\theta)\cos\theta \frac{e^{i(k_0\rho - \pi/4)}}{\sqrt{k_0\rho}} \qquad (12.5a)$$

$$H_{\theta 0}^{(s)} = \frac{i\omega\epsilon_0\zeta\kappa K}{\sqrt{2\pi}} G_{\theta 0}(\theta)\cos\theta \frac{e^{i(k_0\rho - \pi/4)}}{\sqrt{k_0\rho}}\sin\theta \qquad (12.5b)$$

$$H_{\rho 0}^{(s)} = 0 \qquad (12.5c)$$

<u>12.1 b Evaluation of the field integrals along the branch cuts</u>--Since σ_1 and σ_2 are the same as in the corresponding problem of the electric line source, the branch points are the same as in Chapter 9 and the formulation of (9.21) applies. Applying this formulation to the field integrals of (12.1), we obtain for the components of the lateral field

$$H_{y0}^{(B_1)} = \frac{i\omega\epsilon_0\zeta K}{\sqrt{2\pi}} M_{y0}^{(1)} \frac{e^{i[k_0\rho\cos(|\theta|-\theta_{B_1}) - \pi/4]}}{[k_0\rho\sin(|\theta| - \theta_{B_1})]^{3/2}} \qquad (12.6a)$$

$$H_{y0}^{(B_2)} = \frac{i\omega\epsilon_0\zeta K}{\sqrt{2\pi}} M_{y0}^{(2)} \frac{e^{i[k_0\rho\cos(|\theta|-\theta_{B_2}) - \pi/4]}}{[k_0\rho\sin(|\theta| - \theta_{B_2})]^{3/2}} \qquad (12.6b)$$

$$M_{y0}^{(1)} = \frac{(\epsilon+\eta)\cos\theta_{B1}\sqrt{\zeta}\sin 2\theta_{B1}}{\sqrt{\zeta+\epsilon+\eta}\, b_{H1}^2}\left[\sqrt{-\tfrac{\eta}{\epsilon}(\epsilon+\eta+\zeta)} + \sqrt{1-(\epsilon+\eta)}\right] \tag{12.7a}$$

$$\cdot\left[ik_0 h b_{H1} - a_{H1} + (1-\zeta)^2\sqrt{1-(\epsilon+\eta)}\, e^{ik_0 h\sqrt{-\eta/\epsilon(\epsilon+\eta+\zeta)}}\right]$$

$$M_{y0}^{(2)} = \frac{(\epsilon-\eta)\cos\theta_{B2}\sqrt{\zeta}\sin 2\theta_{B1}}{\sqrt{\zeta+\epsilon-\eta}\, b_{H2}^2}\left[\sqrt{\tfrac{\eta}{\epsilon}(\epsilon-\eta+\zeta)} + \sqrt{1-(\epsilon-\eta)}\right] \tag{12.7b}$$

$$\cdot\left[ik_0 h b_{H2} - a_{H2} + (1-\zeta)^2\sqrt{1-(\epsilon-\eta)}\, e^{ik_0 h\sqrt{\eta/\epsilon(\epsilon-\eta+\zeta)}}\right]$$

$$a_{H1,2} = \sqrt{\mp\eta/\epsilon\,(\zeta+\epsilon\pm\eta)}\left[1-(\zeta+\epsilon\pm\eta)\right] - \sqrt{1-(\epsilon\pm\eta)}\left[\zeta^2+(\epsilon\pm\eta-\zeta)\right] \tag{12.7c}$$

$$b_{H1,2} = \sqrt{1-(\epsilon\pm\eta)}\left\{\left[1-\zeta(1+\epsilon\pm\eta)\right]\sqrt{\mp\eta/\epsilon\,(\zeta+\epsilon\pm\eta)} + \zeta\left[1-\zeta-(\epsilon\pm\eta)\right]\sqrt{1-(\epsilon\mp\eta)}\right\} \tag{12.7d}$$

$$\theta_{B1} = \text{arc sin } \sqrt{\epsilon+\eta} \tag{12.7e}$$

$$\theta_{B2} = \text{arc sin } \sqrt{\epsilon-\eta}\,. \tag{12.7f}$$

Furthermore,

$$H_{\theta 0\,\theta\gtrless 0}^{(1)} = \pm\frac{\omega\epsilon_0\zeta K}{\sqrt{2\pi}} M_{\theta 0}^{(1)} \frac{\cos(|\theta|-\theta_{B1})\, e^{i[k_0\rho\cos(|\theta|-\theta_{B1})-\pi/4]}}{[k_0\rho\sin(|\theta|-\theta_{B1})]^{3/2}} \tag{12.8a}$$

$$H_{\theta 0\,\theta\gtrless 0}^{(2)} = \mp\frac{\omega\epsilon_0\zeta K}{\sqrt{2\pi}} M_{\theta 0}^{(2)} \frac{\cos(|\theta|-\theta_{B2})\, e^{i[k_0\rho\cos(|\theta|-\theta_{B2})-\pi/4]}}{[k_0\rho\sin(|\theta|-\theta_{B2})]^{3/2}} \tag{12.8b}$$

$$H_{\rho 0}^{(1)} = +\frac{\omega\epsilon_0\zeta K}{\sqrt{2\pi}} M_{\theta 0}^{(1)} \frac{\sin(|\theta|-\theta_{B1})\, e^{i[k_0\rho\cos(|\theta|-\theta_{B1})-\pi/4]}}{[k_0\rho\sin(|\theta|-\theta_{B1})]^{3/2}} \tag{12.8c}$$

$$H_{\rho 0}^{(2)} = \frac{-\omega\epsilon_0\zeta K}{\sqrt{2\pi}} M_{\theta 0}^{(2)} \frac{\sin(|\theta|-\theta_{B2})\, e^{i[k_0\rho\cos(|\theta|-\theta_{B2})-\pi/4]}}{[k_0\rho\sin(|\theta|-\theta_{B2})]^{3/2}} \tag{12.8d}$$

where

$$M_{\phi o}^{(1)} = \frac{\sqrt{\epsilon+\eta}\cos\theta_{B1}\sqrt{\zeta}\sin 2\theta_{B1}}{\sqrt{\zeta+\epsilon+\eta}\, b_{M1}^{2}} \left[\sqrt{-\kappa(\epsilon+\eta+\zeta)} + \zeta\sqrt{1-(\epsilon+\eta)}\right] \tag{12.9a}$$

$$\cdot\left[ik_o h b_{M1} - a_{M1} + (1-\zeta)^2\sqrt{1-(\epsilon+\eta)}\exp(ik_o h\sqrt{-\kappa(\epsilon+\eta+\zeta)})\right]$$

$$M_{\phi o}^{(2)} = \frac{\sqrt{\epsilon-\eta}\cos\theta_{B2}\sqrt{\zeta}\sin 2\theta}{\sqrt{\zeta+\epsilon-\eta}\, b_{M2}^{2}} \left[\sqrt{\kappa(\epsilon-\eta+\zeta)} + \zeta\sqrt{1-(\epsilon-\eta)}\right] \tag{12.9b}$$

$$\cdot\left[ik_o h b_{M2} - a_{M2} + (1-\zeta)^2\cdot\sqrt{1-(\epsilon-\eta)}\exp(ik_o h\sqrt{\kappa(\epsilon-\eta+\zeta)})\right].$$

12.2 POWER FLOW

In the radiation field all components of the electric and magnetic field are transverse to the radial direction. Therefore, the Poynting vector has only a radial component given by

$$S_{\rho o} = \frac{Z_o}{2}\left\{ |H_{yo}|^2 + |H_{\phi o}|^2 \right\}. \tag{12.10}$$

Now, using the results of (12.5) the above result can be written more explicitly as follows:

$$S_{\rho o} = \frac{\omega\,\epsilon_o\zeta^2 K^2\cos^2\theta}{4\pi\rho}\left\{ |G_{yo}(\theta)|^2 + \kappa^2\sin^2\theta\, |G_{\phi o}(\theta)|^2 \right\} \tag{12.11}$$

where $G_{yo}(\theta)$ and $G_{\phi o}(\theta)$ are given by (12.2).

12.3 CLOSURE

In this chapter we found the field components in the air-region due to a magnetic current line source in a magnetoplasma when the steady magnetic field is normal to the line source. As in the corresponding case of an electric current line source in Chapter 9, the field in the air-region consists of two main components, the radiation field and the lateral field.

The radiation field contains, in addition to the components present in the corresponding isotropic case, additional transverse components of the electric and magnetic fields which render the resulting fields to be elliptically polarized everywhere except right above the source.

CHAPTER 13

THE STEADY MAGNETIC FIELD PARALLEL TO THE LINE SOURCE

In this chapter we shall be concerned with finding the electric and magnetic fields due to a magnetic line source situated in a magnetoplasma when the steady magnetic field is along the direction of the line source.

13.1 FORMULATION OF THE PROBLEM

The geometry of the problem is shown in Fig. 13.1. The horizontal plane $z = 0$ coincides with the interface between the anisotropic homogeneous plasma and air. For convenience we shall call the plasma medium (1) and the air medium (0). As before, we assume that both media have the same magnetic inductive capacity of free space, μ_0 . The steady magnetic field will, in this case, be oriented along the line source whereas the line source is parallel to the interface.

<u>13.1 a Fundamental equations</u>—The solution to the present boundary value problem entails solution to the Maxwell's equations in the following form:

$$\begin{aligned}
\vec{\nabla} \times \vec{E}_1 &= i\omega\mu_0 \vec{H}_1 - K\,\delta(y)\,\delta(z+h)\,\vec{1}_x \\
\vec{\nabla} \times \vec{H}_1 &= -i\omega\epsilon_0 \overline{\overline{\epsilon}}\,\vec{E}_1 \\
\vec{\nabla} \cdot \vec{D}_1 &= 0 \\
\vec{\nabla} \cdot \vec{B}_1 &= \rho_m
\end{aligned} \qquad (13.1)$$

where K, as before, denotes the magnetic current and ρ_m , the magnetic charge.

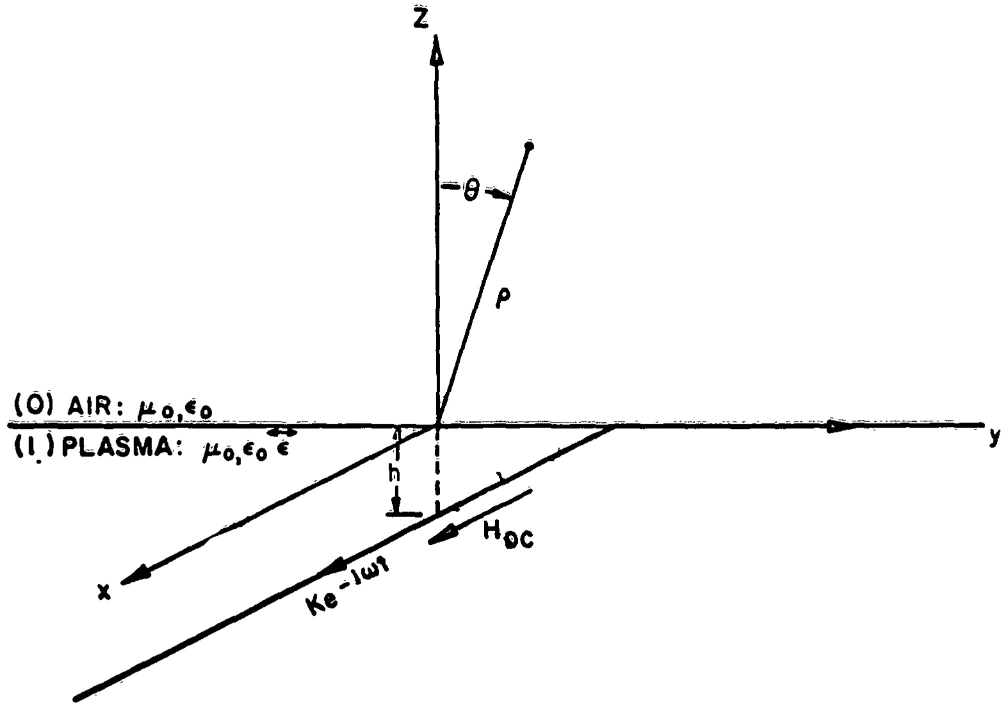

Figure 13.1 - Geometry of the Problem of A Magnetic Current Line Source When The Steady Magnetic Field is Parallel

In the air-region the fields must satisfy

$$
\begin{aligned}
\nabla \times \vec{E}_o &= i\omega\mu_o \vec{H}_o \\
\nabla \times \vec{H}_o &= -i\omega\mu_o \vec{E}_o \\
\nabla \cdot \vec{D}_o &= 0 \\
\nabla \cdot \vec{B}_o &= 0
\end{aligned}
\tag{13.2}
$$

Moreover, our solution must insure proper behavior of the fields at infinity as well as the continuity of the tangential components of the fields at the interface.

<u>13.1 b The integral representation of the field components in the plasma</u>—We form the wave equation by performing a curl operation on the second equation of (13.1) and then substituting the first. This gives

$$
-\nabla \times \overleftrightarrow{\epsilon}^{-1} \nabla \times \vec{H}_1 + k_o^2 \vec{H}_1 = -i\omega\mu_o K \,\delta(y)\,\delta(z+h)\,\vec{1}_x \tag{13.3}
$$

Noting that $\partial_x = 0$ and using the components of the inverse of the permittivity tensor from equation (2.9), we obtain

$$
\begin{bmatrix} \chi k_o^2 + \partial_y^2 + \partial_z^2 & 0 & 0 \\ 0 & \zeta k_o^2 + \partial_z^2 & -\partial_y\partial_z \\ 0 & -\partial_y\partial_z & \zeta k_o^2 + \partial_y^2 \end{bmatrix} \begin{bmatrix} H_{x1} \\ H_{y1} \\ H_{z1} \end{bmatrix} = -i\omega\epsilon_o\chi K\,\delta(y)\,\delta(z+h) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{13.4}
$$

From (13.4) it appears that a single component of the magnetic field may be sufficient to solve the present boundary value problem. The corresponding differential equation is

$$
(\chi k_o^2 + \partial_y^2 + \partial_z^2)\, H_{x1} = -i\omega\epsilon_o\chi K\,\delta(y)\,\delta(z+h) \tag{13.5}
$$

For convenience we introduce the Fourier transform pair

$$\widetilde{H}_{x1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_{x1}\, e^{-i\alpha_2 y}\, dy$$

$$H_{x1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{H}_{x1}\, e^{i\alpha_2 y}\, d\alpha_2 \tag{13.6}$$

whereupon we transform equation (13.5) as follows:

$$(\partial_z^2 + s_1^2)\, \widetilde{H}_{x1} = -\frac{i\omega\epsilon_0 \chi K}{\sqrt{2\pi}}\, \delta(z+h) \tag{13.7}$$

since the vanishing of the integrated part is assured when the radiation condition is satisfied. The parameter s_1 is given by

$$s_1 = \sqrt{\chi k_0^2 - \alpha_2^2}$$
$$\text{Im}\{s_1\} > 0\,. \tag{13.8}$$

The solution to the differential equation (13.7) is well-known and we write it at once

$$\widetilde{H}_{x1}^{(P)} = -\frac{\omega\epsilon_0 \chi K\, e^{is_1|z+h|}}{2\sqrt{2\pi}\, s_1} \tag{13.9}$$

Inverting with respect to the α_2 transform variable, we obtain the particular integral

$$H_{x1}^{(P)} = -\frac{\omega\epsilon_0 \chi K}{4\pi} \int_{-\infty}^{\infty} \frac{e^{i(\alpha_2 y + s_1|z+h|)}}{s_1}\, d\alpha_2 \tag{13.10}$$

The form of the complementary field in the plasma immediately suggests itself. We write

$$H_{x1}^{(c)} = -\frac{\omega\epsilon_0 \chi K}{4\pi} \int_{-\infty}^{\infty} A_1\, e^{i(\alpha_2 y - s_1 z)}\, d\alpha_2 \tag{13.11}$$

We also note that the electric field components can be found from (2.27) with $\partial_x = 0$. We get

$$\begin{bmatrix} E_{x_1} \\ E_{y_1} \\ E_{z_1} \end{bmatrix} = \frac{1}{i\omega\epsilon_0\chi} \begin{bmatrix} 0 & \frac{\chi}{\xi}\partial_z & -\frac{\chi}{\xi}\partial_y \\ -i\kappa\partial_y - \partial_z & 0 & 0 \\ \partial_y - i\kappa\partial_z & 0 & 0 \end{bmatrix} \begin{bmatrix} H_{x_1} \\ H_{y_1} \\ H_{z_1} \end{bmatrix} \tag{13.12}$$

<u>13.1 c The integral representation of the field components in the air</u>— In the air the magnetic field components satisfy the homogeneous wave equation

$$(\nabla^2 + k_0^2)\vec{H}_0 = 0. \tag{13.13}$$

Thus, we can write

$$H_{x_0} = -\frac{\omega\epsilon_0\chi K}{4\pi}\int_{-\infty}^{\infty} A_0 e^{i(\alpha_2 y + s_0 z)} d\alpha_2 \tag{13.14}$$

<u>13.1 d The boundary conditions</u>—The requirement of the continuity of the tangential field components at the boundary $z = 0$ amounts to satisfying the following equations:

$$\begin{aligned} H_{x_0} &= H_{x_1} \\ \chi\partial_z H &= (i\kappa\partial_y + \partial_z)H_{x_1} \end{aligned} \tag{13.15}$$

Upon performing the indicated differentiation, we obtain two simultaneous equations in two unknowns

$$\begin{bmatrix} \chi s_0 & s_1 - i\kappa\alpha_2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} = \frac{e^{is_1h}}{s_1} \begin{bmatrix} s_1 + i\kappa\alpha_2 \\ 1 \end{bmatrix} \tag{13.16}$$

from which we obtain

$$A_0 = \frac{2e^{is_1h}}{s_1 + (\chi s_0 - i\kappa\alpha_2)} \tag{13.17a}$$

$$A_1 = \frac{2s_1 e^{is_1h}}{s_1 + (\chi s_0 - i\kappa\alpha_2)} - \frac{e^{is_1h}}{s_1} \tag{13.17b}$$

Substituting the above results in the corresponding integrals, we obtain the

integral representation of the fields as follows:

$$H_{x_1} = -\frac{\omega\epsilon_0\chi K}{4\pi}\int_{-\infty}^{\infty}\left\{\frac{e^{i(\alpha_2 y + s_1|z+h|)}}{s_1} - \frac{e^{i[\alpha_2 y - s_1(z-h)]}}{s_1} + \frac{2e^{i[\alpha_2 y - s_1(z-h)]}}{s_1 + (\chi s_0 - i\kappa\alpha_2)}\right\} d\alpha_2 \tag{13.18a}$$

$$H_{x_0} = -\frac{\omega\epsilon_0\chi K}{2\pi}\int_{-\infty}^{\infty}\frac{e^{is_1 h}}{s_1 + \chi s_0 - i\kappa\alpha_2}\, e^{i(\alpha_2 y + s_0 z)}\, d\alpha_2 \tag{13.18b}$$

13.2 THE FIELD COMPONENTS

Having found the integral representation of the field components in each region, we shall now reduce them to the form suitable for numerical computations. In this process we shall avail ourselves of the techniques of saddle-point and branch-cut integration formulated in the previous chapters.

13.2 a The fields in the plasma—We transform the integral representation (13.18a) to cylindrical coordinates in both transform and the coordinate spaces using

$$\begin{aligned} \alpha_2 &= \sqrt{\chi}\,\kappa_0 \sin\beta \\ y &= \rho_{1,2}\sin\theta_{1,2} \\ z + h &= \rho_1\cos\theta_1 \\ z - h &= \rho_2\cos\theta_2 \end{aligned} \tag{13.19}$$

and obtain

$$H_{x_1} = -\frac{\omega\epsilon_0\chi K}{4\pi}\int_{\Gamma_1}\left\{ e^{i\sqrt{\chi}k_0\rho_1\cos(\beta-\theta_1)} - e^{i\sqrt{\chi}k_0\rho_2\cos(\beta-\theta_2)} + \frac{2\cos\beta\, e^{i\sqrt{\chi}k_0\rho_2\cos(\beta-\theta_2)}}{\cos\beta + \chi\sqrt{\frac{1}{\chi} - \sin^2\beta} - i\kappa\sin\beta}\right\} d\beta\,. \tag{13.20}$$

The first two terms in the above expression can be recognized as integral representations of Hankel functions. Thus, we write

$$H_{x_1} = -\frac{\omega\epsilon_0\chi K}{4}\left\{H_0^{(1)}(\sqrt{\chi}\,k_0\rho_1) - H_0^{(1)}(\sqrt{\chi}\,k_0\rho_2)\right\} + H_{x_1}^{(r)} \qquad (13.21a)$$

where

$$H_{x_1}^{(r)} = \frac{+\omega\epsilon_0\chi K}{2\pi}\int_{\Gamma}\frac{\cos\beta\, e^{i\sqrt{\chi}\,k_0\rho_2\cos(\beta-\theta)}\,d\beta}{\cos\beta + \chi\sqrt{\frac{1}{\chi} - \sin^2\beta} - i\kappa\sin\beta} \qquad (13.21b)$$

The integrand in (13.21b) contains the radical $(\frac{1}{\chi} - \sin^2\beta)^{\frac{1}{2}}$ as a result of which the point $\beta = \theta_B$ where

$$\theta_B = \pm \text{arc}\sin\left(\frac{1}{\sqrt{\chi}}\right) \qquad (13.22)$$

will be a branch point. At each point β the integrand in (13.21b) can take on two values depending on which sign we choose for the radical. As before, it is convenient here to talk about two sheets of the β-plane (formed by a two-sheeted Riemann surface) on which the integrand is single valued. The two sheets will be joined along the lines

$$\text{Im}\left\{\sqrt{\frac{1}{\chi} - \sin^2\beta}\right\} = 0 \qquad (13.23)$$

starting at the branch point.

As before, the result will consist of the saddle-point contribution and the contribution from the branch cut representing the radiation field and the lateral field respectively. Thus, we write

$$H_{x_1}^{(r)} = H_{x_1}^{(rs)} + H_{x_1}^{(rB)}\,u(\theta - \theta_B) \qquad (13.24)$$

and evaluate each one of these contributions separately.

To find the saddle-point contribution we use (9.12) and obtain

$$H_{x_1}^{(rs)} \sim \frac{+\omega\epsilon_0\chi K}{\sqrt{2\pi}}\cdot\frac{\cos\theta}{\cos\theta + \chi\sqrt{\frac{1}{\chi} - \sin^2\theta} - i\kappa\sin\theta}\;\frac{e^{i(\sqrt{\chi}\,k_0\rho_2 - \pi/4)}}{\sqrt{\sqrt{\chi}\,k_0\rho_2}} \qquad (13.25)$$

The branch-cut contribution was formulated in (9.21a). Using this formulation

we can write

$$H_{x_1}^{(rB)} = i e^{i\sqrt{\chi}k_o\rho_2\cos(|\theta|-\theta_B)} \int_0^\infty x R(\pm\beta) e^{-\sqrt{\chi}k_o\rho_2\sin(|\theta|-\theta_B)x^2/2} dx \tag{13.26}$$

where

$$R(\pm\beta) = F_-(\pm\beta) - F_+(\pm\beta) \tag{13.27}$$

and

$$\beta = \theta_B + \frac{i x^2}{2} .$$

Now we find

$$R(\pm\beta) = \frac{\omega\epsilon_o\chi^2 K}{\pi} \cdot \frac{\cos\beta\sqrt{\frac{1}{\chi} - \sin^2\beta}}{(\cos\beta \mp i\kappa\sin\beta)^2 - \chi^2(\frac{1}{\chi} - \sin^2\beta)} . \tag{13.28}$$

We can set $\beta = \theta_B$ everywhere except in the radical which becomes zero at that point. We compute it then to the next higher approximation. We obtain

$$\sqrt{\frac{1}{\chi} - \sin^2\beta} \sim -e^{-i\pi/4}\sqrt[4]{\frac{\chi-1}{\chi^2}}\, x \tag{13.29}$$

where of the two possible signs of the radical we have chosen the one that makes $\mathrm{Im}\left\{\sqrt{\frac{1}{\chi} - \sin^2\beta}\right\} \geqslant 0$. We find for the leading term

$$R(\pm\beta) \sim -\frac{\omega\epsilon_o\chi^2 K e^{-i\pi/4}(\chi-1)^{3/4}}{\pi(\sqrt{\chi-1} \mp i\kappa)^2}\, x . \tag{13.30}$$

Substituting the above result into (3.26) and performing the integration where we use the result of (4.31b), we obtain finally

$$H_{x_1}^{(rB)} = \frac{-\omega\epsilon_o\chi^2 K(\chi-1)^{3/4}\exp\{i[\sqrt{\chi}k_o\rho_2\cos(|\theta| - \theta_B) + \pi/4]\}}{\sqrt{2\pi}\,(\sqrt{\chi-1} \mp i\kappa)^2\left[\sqrt{\chi}k_o\rho_2\sin(|\theta| - \theta_B)\right]^{3/2}} . \tag{13.31}$$

We observe that the lateral wave in this case is not symmetric as before but it is still of second order.

To summarize the above results we write for the first order magnetic field in the plasma

$$H_{x_1} = \frac{-\omega\epsilon_0\sqrt{\chi}\,K e^{-i\pi/4}}{\sqrt{2\pi k_0}}\left\{\frac{e^{i\sqrt{\chi}k_0\rho_1}}{\sqrt{\rho_1}} - \frac{\cos\theta-(\chi\sqrt{1/\chi-\sin^2\theta}-i\kappa\sin\theta)}{\cos\theta-i\kappa\sin\theta+\chi\sqrt{1/\chi-\sin^2\theta}}\;\frac{e^{i\sqrt{\chi}k_0\rho_2}}{\sqrt{\rho_2}}\right\}. \tag{13.32}$$

The electric field components in the plasma can be found from (13.12). We write them in cylindrical coordinates

$$\begin{aligned} E_{\theta_1} &= -\frac{Z_0}{\sqrt{\chi}}\,H_{x_1} \\ E_{\rho_1} &= -\frac{i\kappa Z_0}{\sqrt{\chi}}\,H_{x_1} \;. \end{aligned} \tag{13.33}$$

We note that the electric field is not purely transverse. However, the displacement vector $\vec{D}$ is transverse. We find

$$\begin{aligned} D_{\theta_1} &= -\epsilon_0\sqrt{\chi}\;Z_0 H_{x_1} \\ D_{\rho_1} &= 0\;. \end{aligned} \tag{13.34}$$

<u>13.2 b The fields in the air</u>—We first transform the integral representation (3.18b) to cylindrical coordinates in both transform and coordinate spaces using

$$\begin{aligned} \alpha_2 &= k_0\sin\beta \\ y &= \rho\sin\theta \\ z &= \rho\cos\theta \end{aligned} \tag{13.35}$$

and obtain

$$H_{x_0} = -\frac{\omega\epsilon_0\chi K}{2\pi}\int_{\Gamma}\frac{\cos\beta\, e^{ik_0h\sqrt{\chi-\sin^2\beta}}}{\sqrt{\chi-\sin^2\beta}+\chi\cos\beta-i\kappa\sin\beta}\,e^{ik_0\rho\cos(\beta-\theta)}\,d\beta\,. \tag{13.36}$$

The integrand contains the radical $(\chi-\sin^2\beta)^{\frac{1}{2}}$ as a result of which the point $\beta=\theta_B$ where

$$\theta_B = \pm\arcsin\sqrt{\chi}$$

will now be the branch point. Following the method of the previous sections, we write

$$H_{xo} = H_{xo}^{(s)} + H_{xo}^{(B)}\, u\,(\theta - \theta_B) \tag{13.37}$$

where $H_{xo}^{(s)}$ and $H_{xo}^{(B)}$ are the saddle-point and branch-cut contributions respectively. The saddle-point contribution can be found immediately using (9.12). We obtain

$$H_{xo}^{(s)} = -\frac{\omega\epsilon_0\chi K}{\sqrt{2\pi k_0\rho}}\cdot\frac{\cos\theta\, e^{ik_0h\sqrt{\chi-\sin^2\theta}}}{\sqrt{\chi-\sin^2\theta}+\chi\cos\theta - i\kappa\sin\theta}\, e^{i(k_0\rho-\pi/4)}. \tag{13.38}$$

To evaluate the branch-cut contribution we use the formulation of equation (9.21a) and write

$$H_{xo}^{(B)} = ie^{ik_0\rho\cos(|\theta|-\theta_B)}\int_0^\infty x R(\pm\beta)e^{-k_0\rho\sin(|\theta|-\theta_B)x^2/2}\, dx \tag{13.39}$$

where

$$R(\pm\beta) = F_-(\pm\beta) - F_+(\pm\beta)$$

and

$$\beta = \theta + \frac{ix^2}{2}. \tag{13.40}$$

Now we find

$$R(\pm\beta) = \frac{-\omega\epsilon_0\chi K\cos\beta\left[\sqrt{\chi-\sin^2\beta}\cos(k_0h\sqrt{\chi-\sin^2\beta})-i(\chi\cos\beta\mp i\kappa\sin\beta)\sin(k_0h\sqrt{\chi-\sin^2\beta})\right]}{\pi\left[(\chi\cos\beta\mp i\kappa\sin\beta)-(\chi-\sin^2\beta)\right]}. \tag{13.41}$$

We set $\beta = \theta_B$ everywhere except in the radical which becomes zero at that point. We compute it then to the next higher approximation and obtain

$$\sqrt{\chi-\sin^2\beta} \sim -e^{-i\pi/4}\sqrt{\chi(1-\chi)}\; x \tag{13.42}$$

where of the two possible choices of the sign for the radical we have chosen the one that makes $\mathrm{Im}\{(\chi-\sin^2\beta)^{1/2}\} > 0$. Substituting the above expression into (13.41), we obtain for the leading term

$$R(\pm\beta) \sim \frac{\omega\epsilon_0 K e^{-i\pi/4}\sqrt[4]{\chi(1-\chi)^3}\left[1-i\sqrt{\chi}\,k_0 h(\sqrt{\chi(1-\chi)}\mp i\kappa)\right]}{\pi\left[\sqrt{\chi(1-\chi)}\mp i\kappa\right]^2}\,x\,. \tag{13.43}$$

Substituting the last result into the integral of (13.39) and performing the integration where we use the results of (4.31b), we obtain finally

$$\underset{\theta\gtrless 0}{H_{x0}^{(B)}} = \frac{\omega\epsilon_0 K\sqrt[4]{\chi(1-\chi)^3}\left[1-i\sqrt{\chi}\,k_0 h(\sqrt{\chi(1-\chi)}\mp i\kappa)\right]e^{i[k_0\varrho\cos(\theta+\theta_B)-\pi/4]}}{\sqrt{2\pi}\left[\sqrt{\chi(1-\chi)}\mp i\kappa\right]^2\left[k_0\varrho\sin(|\theta|-\theta_B)\right]^{3/2}}\,. \tag{13.44}$$

We note that the lateral wave in the air, when it exists, is not symmetric but it is still of second order.

The electric field components in the air can be found from the appropriate curl equation. The only non-vanishing component is $E_{\theta 0}$ and is given by

$$E_{\theta 0} = -Z_0 H_{x0} \tag{13.45}$$

to the first order.

The Poynting vector has only a radical component and is given by

$$S_\varrho = \frac{\omega\epsilon_0\chi^2K^2}{4\pi\varrho}\left|\frac{\cos\theta\, e^{ik_0h\sqrt{\chi-\sin^2\theta}}}{\sqrt{\chi-\sin^2\theta}+\chi\cos\theta-i\kappa\sin\theta}\right|^2 \tag{13.46}$$

Examination of (13.46) reveals some interesting possibilities. For when $\chi > \sin^2\theta$ one can write for a lossless plasma,

$$S_\varrho = \frac{\omega\epsilon_0\chi^2K^2}{4\pi\varrho}\cdot\frac{\cos^2\theta}{(\sqrt{\chi-\sin^2\theta}+\chi\cos\theta)^2+\kappa^2\sin^2\theta} \tag{13.47a}$$

and when $\chi < \sin^2\theta$

$$S_\varrho = \frac{\omega\epsilon_0\chi^2K^2}{4\pi\varrho}\cdot\frac{\cos^2\theta\, e^{-2k_0h\sqrt{\sin^2\theta-\chi}}}{\chi^2\cos^2\theta+(\sqrt{\sin^2\theta-\chi}-\kappa\sin^2\theta)^2} \tag{13.47b}$$

Then it follows that for $\chi > \sin^2\theta$ the power pattern will be symmetric about the axis $\theta = 0$ and unaffected by the depth of the source burial, h. When $\chi < \sin^2\theta$ the opposite is true; the pattern, in general, will not be symmetric and its magnitude will decrease with the increased depth of the

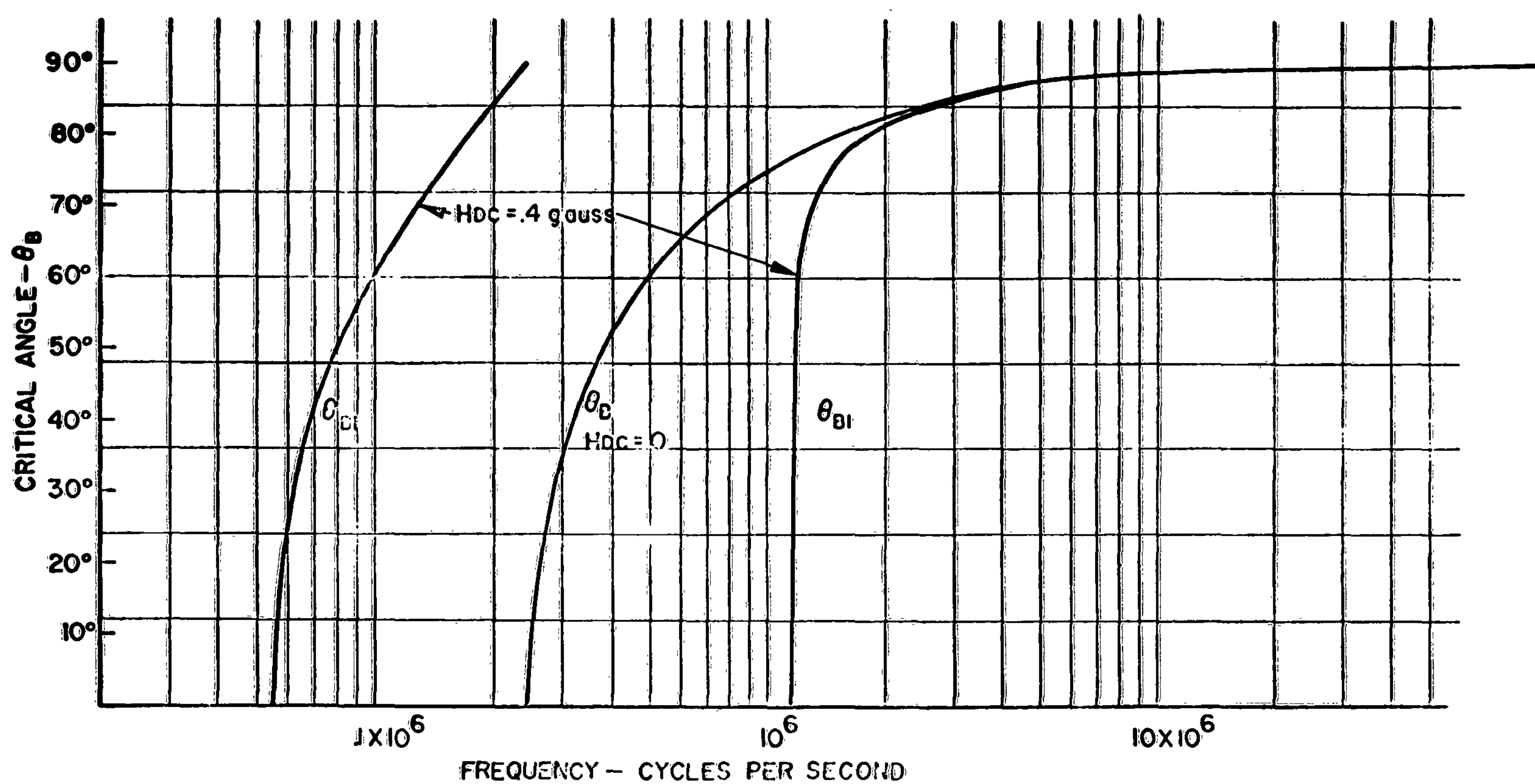

Figure 13.2 - Critical Angles Pertinent to the Problem of a Magnetic Current Line Source When H_{DC} is Parallel

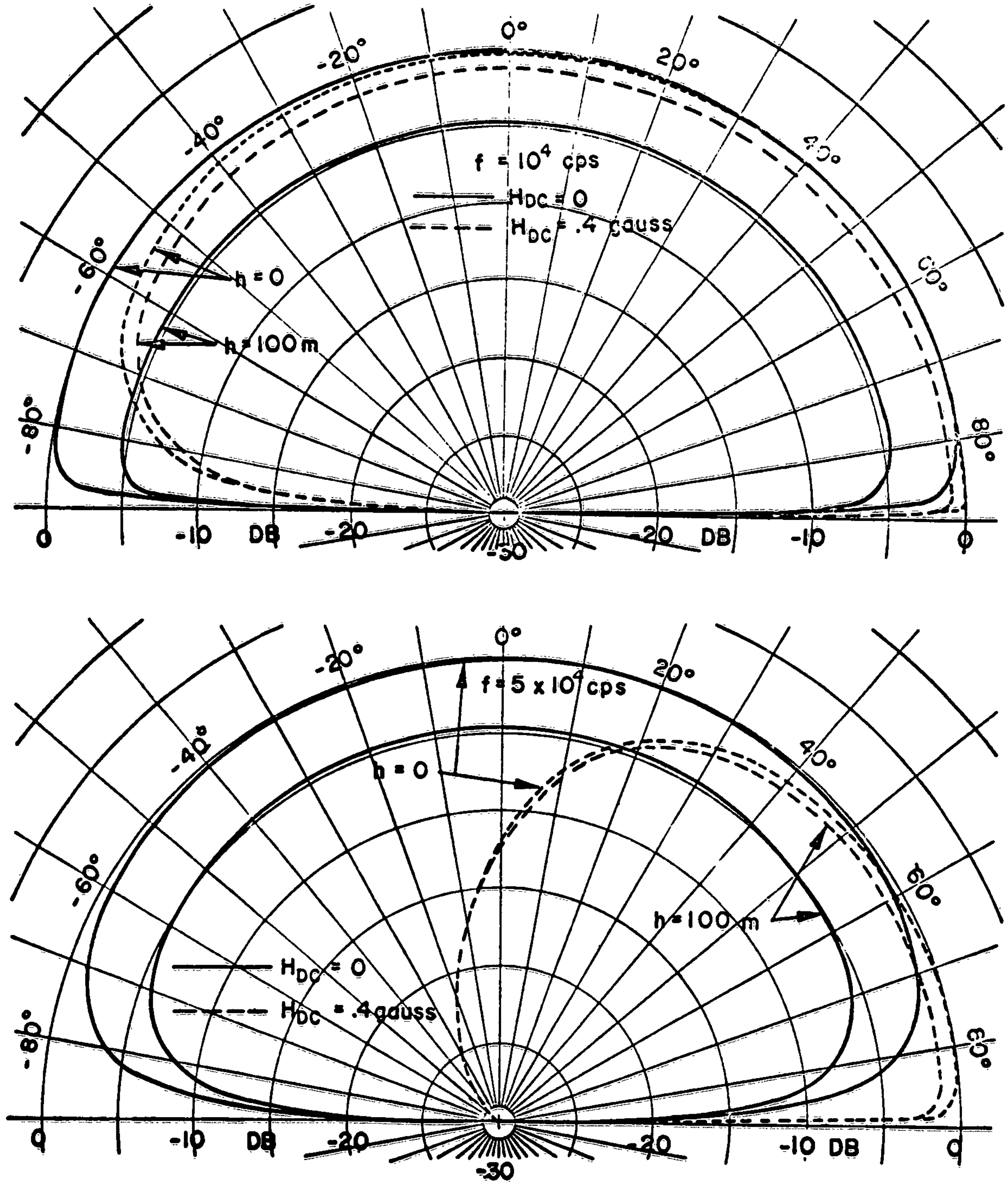

Figure 13.3 - Power Pattern in Air of a Magnetic Current Line Source in Magnetoplasma; H_{DC} along the Line Source, N = 750 electrons per cubic centimeter, h = 0, 100 meters, H_{DC} = 0, .4 gauss, f = 10^4, 5 x 10^4 cycles per second

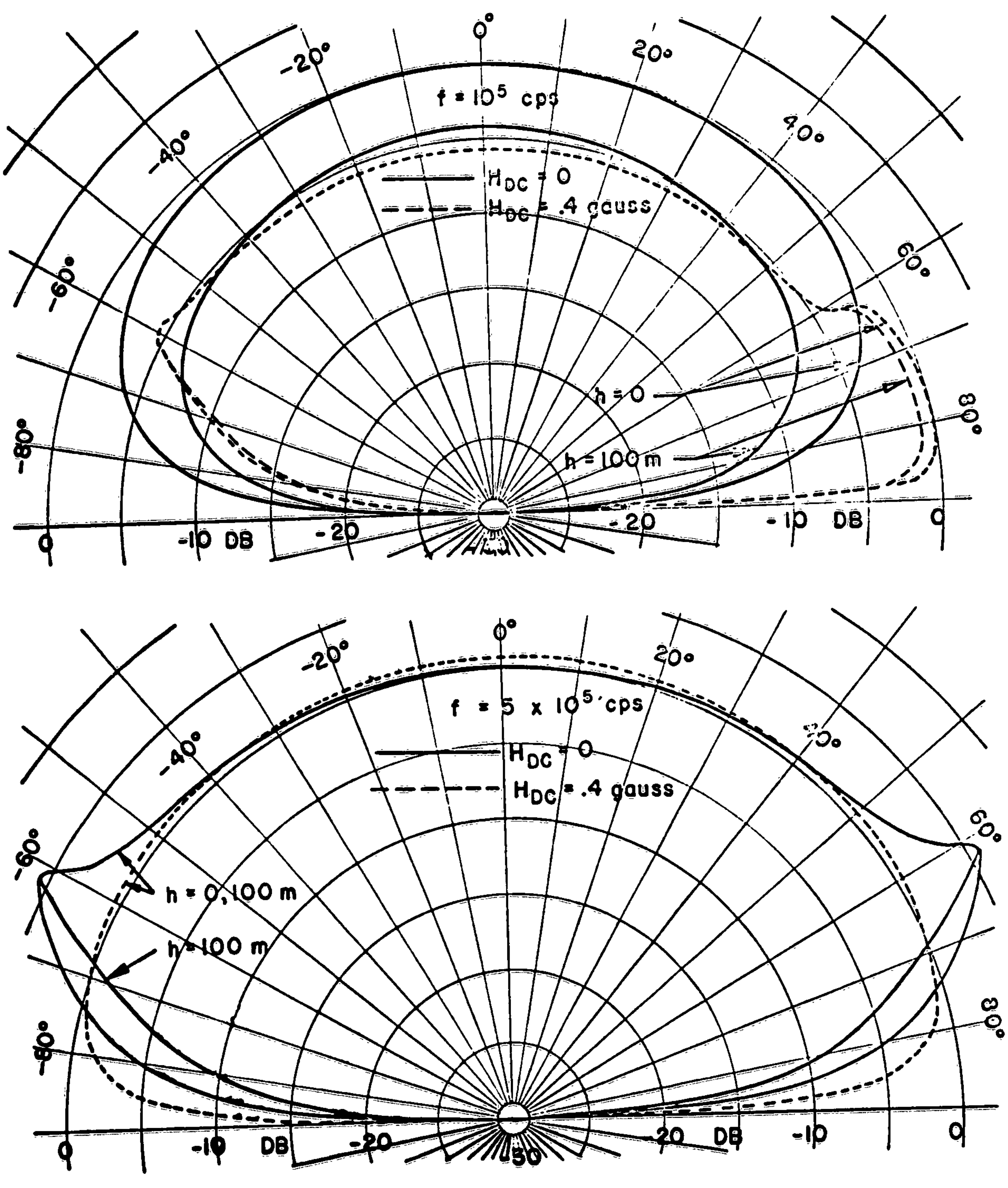

Figure 13.4 - Power Pattern in Air of a Magnetic Current Line Source in Magnetoplasma; H_{DC} along the Line Source, N = 750 electrons per cubic centimeter, h = 0, 100 meters, H_{DC} = 0, .4 gauss, f = 10^5, 5×10^5 cycles per second

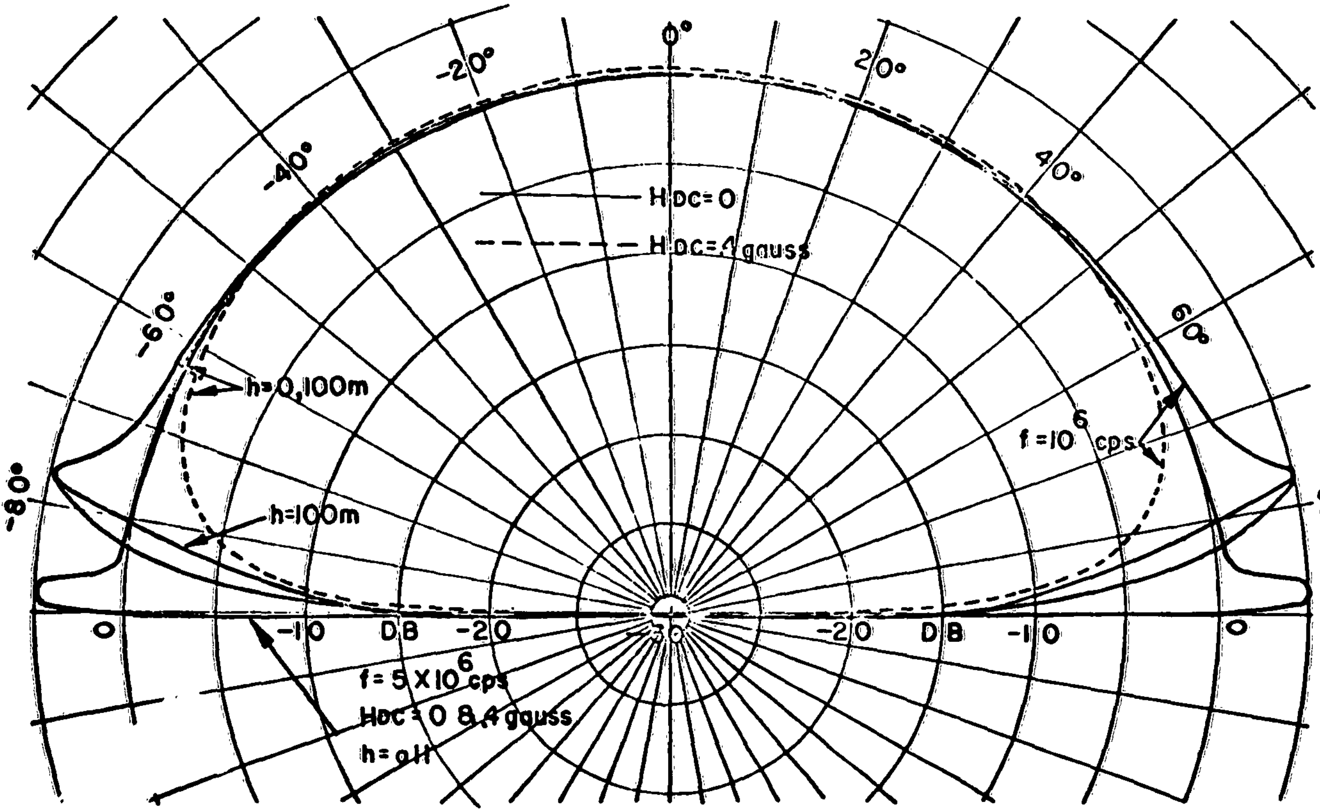

Figure 13.5 - Power Pattern in Air of a Magnetic Current Line Source in Magnetoplasma; H_{DC} Along the the Line Source, N = 750 electrons per cubic centimeter, h = 0, 100 meters, $H_{DC} = 0$, .4 gauss, $f = 10^6$, 5×10^6 cycles per second

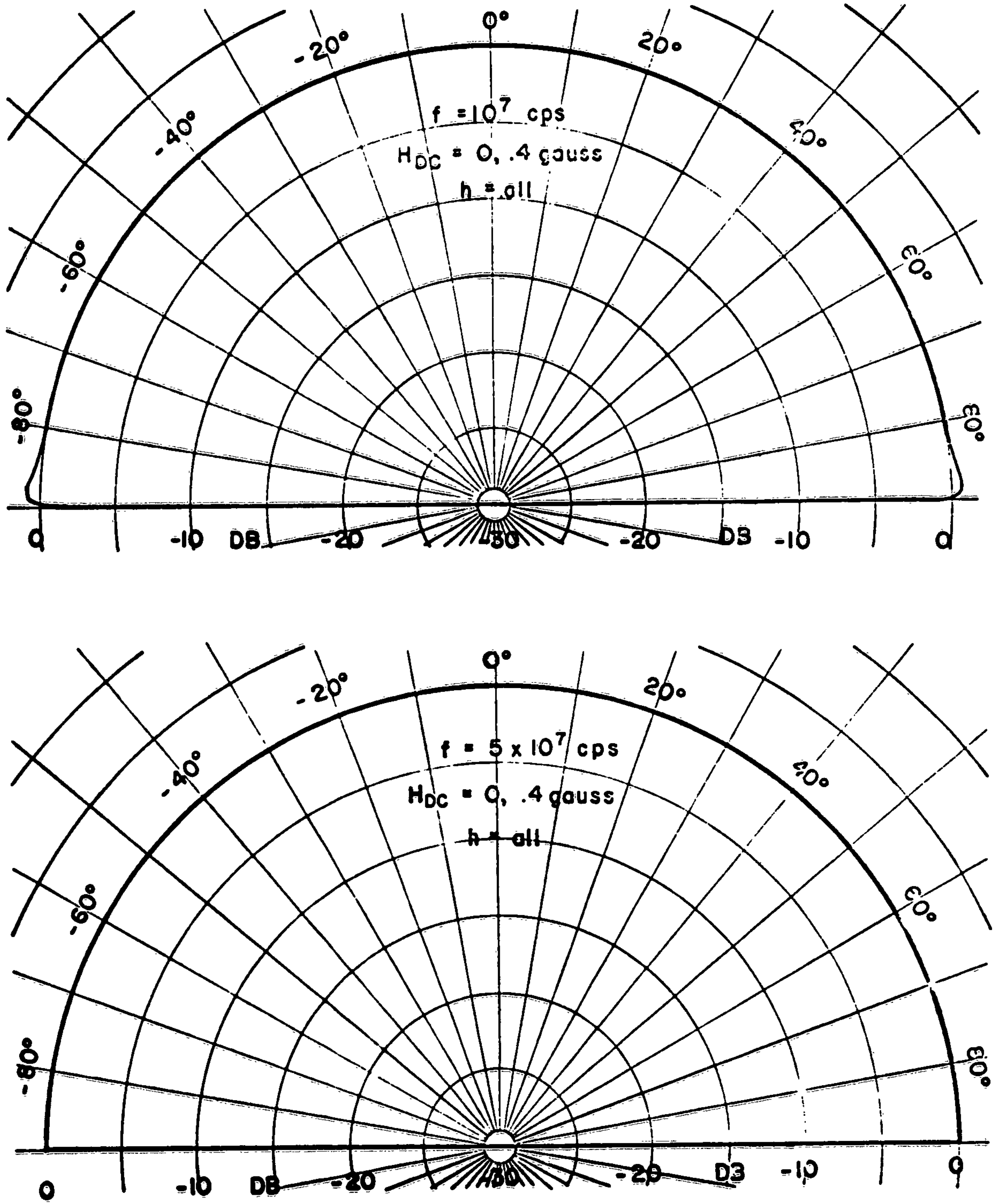

Figure 13.6 - Power Pattern in Air of a Magnetic Current Line Source in Magnetoplasma; H_{DC} along the Line Source, N = 750 electrons per cubic centimeter, h = all, H_{DC} = 0, .4 gauss, f = 10^7, $5x10^7$ cycles per second

source burial, h. Moreover, we note that the reversal of the direction of the steady magnetic field which changes the sign of "κ" has the same effect as the reversing of the sign of the angle θ. Thus, the pattern corresponding to $-H_{DC}$ has a mirror symmetry with respect to the one corresponding to H_{DC}.

To illustrate the above conclusions a numerical example was considered. It was assumed that the source is situated in the lower edge of the ionosphere which, for the purpose of this example, was assumed to be homogeneous and sharply bounded, having an electron density $N = 750$ electrons per cubic centimeter and the earth's magnetic field $H_{DC} = .4$ gauss. The power patterns for the depth of burial of 100 meters and zero meters for several frequencies are shown in Figures 13.3 - 13.6 where, for comparison also, the power patterns corresponding to $H_{DC} = 0$ were superimposed.

The physical interpretation of these patterns is as follows. For polar angles smaller than the critical angle, the transmitted rays undergo the usual refraction fixed by Snell's law for real angles

$$\sqrt{\chi}\sin\theta_1 = \sin\theta \qquad (13.48)$$

where θ_1 is the incidence angle in the magnetoplasma. Since the radiation from the line source itself is symmetric, the resultant pattern is also symmetric within the wedge region $-\theta_B < \theta < \theta_B$. Beyond the critical angle the transmitted energy is produced by the inhomogeneous waves which are also being emanated from the source. A combined action of the interface and the steady magnetic field (see equations (13.15) and (13.17) destroys the symmetry of the inhomogeneous wave amplitudes in the magnetoplasma which then results in lack of symmetry of the power patterns in the air in the regions beyond the critical angle. An excellent correlation between the values of the critical angle θ_B in Fig. 13.2 and the power patterns in Figures 13.3 - 13.5 can be noted. The critical angle θ_B does not have a real value until the frequency reaches

about 52×10^3 cycles per second. Thus, the patterns in Fig. 13.3 and corresponding to frequencies 10^4 and 5×10^4 cycles per second are not symmetric since they are solely produced by the inhomogeneous waves emanating from the source. At a frequency of 10^5 cycles per second, the critical angle $\theta_s = 61$ degrees and it is apparent from the upper pattern of Fig. 13.4 that this is the point where the non-symmetry of the pattern begins.

Finally, we mention about the effect of the depth of the source burial. It is apparent, especially at low frequencies that the transmission through the ionosphere is improved considerably due to the action of the steady magnetic field. Whereas the magnitude of the transmitted power diminishes with the increased depth of the source's burial if no steady magnetic field is present, it is almost independent of it when the steady magnetic field is present. This is due to the changes produced by the steady magnetic field in the effective dielectric constant of the plasma.

13.3 CLOSURE

In this chapter we have formulated and solved the problem of a magnetic current line source situated in a magnetoplasma when the steady magnetic field is along the line source and the line source is parallel to the boundary. In particular, we found the radiation field in the air and in the magnetoplasma as well as the lateral field in each region when the conditions are favorable for its existence.

The resultant radiation pattern in air are not symmetric at lower frequencies but they become symmetric with increasing frequency. Moreover, we found the presence of the steady magnetic field, even of small magnitude, will improve the transmission through the plasma a great deal in comparison with the case of no steady magnetic field.

CHAPTER 14

CONCLUSIONS

In the preceding chapters we have shown the results of successful analyses of several cases of radiation from sources in the presence of an anisotropic plasma half-space.

In Part I which includes Chapters 2 - 7, we were concerned with the problem of a horizontal magnetic dipole in and out of magnetoplasma when the steady magnetic field is parallel to the axis of the dipole. The rigorous formulation of the problem was carried out to the point where the determination of the pertinent boundary coefficients remained to be a straight-forward, but not simple, algebraic process. Due to the prohibitive algebraic complexity involved in the explicit finding of tne necessary boundary coefficients, the high frequency approximation was introduced and the boundary coefficients pertinent to the air-region subsequently found. In this process it was found possible to separate the contributions of the plasma's anisotropy explicitly. The radiation field in the air-region was found within the restrictions of the high frequency approximation but the lateral field could not be evaluated in this formulation due to the fact that the main contribution to the lateral field comes from the point where the high frequency approximation is not valid. The result of this analysis was then applied to the problem of a horizontal magnetic dipole situated near the lower edge in the ionosphere. It appeared that in the region of wave frequencies at which the high frequency

approximation is valid, the correction due to the earth's magnetic field manifests itself most strongly in the polar plane through the axis of the dipole and $\pi/4$ radians from it. Furthermore, this correction appears to be significant only for large polar angles, i.e., in the regions close to the interface.

Part II which consists of Chapters 8 - 10 is devoted to the problems of electric current line sources in the magnetoplasma half-space. The case of the steady magnetic field normal to the line source is rigorously formulated in Chapter 8 and the field integrals pertaining to the air-region are evaluated in Chapter 9 with no restriction on the wave frequency. The radiation field contains, in addition to the components present in the corresponding isotropic case, additional transverse components of the electric and magnetic field which render the resulting fields to be eliptically polarised everywhere except right above the source. The lateral field is of second order as in the corresponding isotropic case, however, unlike in that case, it may consist of two waves rather than one. These waves propagate along the interface with phase velocities characteristic of the right- and left-hand circularly polarized plane waves in the magnetoplasma. From the numerical example considered it is concluded that the transmission of energy from the electric current line source through the plasma and into the free space is generally enhanced by the presence of the steady magnetic field especially at wave frequencies below plasma frequency. At wave frequencies well above the plasma frequency the action of the steady magnetic field becomes insignificant which is not at all surprising. Moreover, the radiation patterns exhibit peaks at points corresponding to the critical angles which in turn correspond to the indices of refraction of the right- and left-hand circularly polarised waves in the magnetoplasma. The remaining chapter of Part II, Chapter 10, is devoted to the formulation and solution of the problem of an electric current line source in

the magnetoplasma when the steady magnetic field is parallel to the line source. In this case it is found that the steady magnetic field has no effect on the radiation field. This is consistent with what one would expect since in this case the electric field is always along the steady magnetic field, thus, the cross-product $\vec{u} \times \vec{B}_{oc}$ vanishes and no interaction with the steady magnetic field takes place.

Part III consisting of Chapters 11 - 13 is devoted to the problems of magnetic current line sources. The case of the steady magnetic field normal to the line source is formulated rigorously in Chapter 11 and the integrals are evaluated in Chapter 12 with no restriction on frequency. This case is analogous to the corresponding case of Part II and the conclusions drawn in that part apply here also. Chapter 13 is devoted to the formulation and solution of the case when the steady magnetic field is along the line source. Here it is found that the magnetoplasma acts as a single-refracting medium which is consistent with physical reasoning since in this case the steady magnetic field is always normal to the alternating electric field, both field vectors being in a plane normal to the direction of propagation. The action of the interface produces, however, a very interesting phenomenon in the form of the transmission coefficient which is not symmetric with respect to the vertical axis through the source. Moreover, the magnitude of the transmission coefficient and the phase may, under certain conditions, depend on the orientation of the steady magnetic field. This produces some rather interesting phenomena in the radiation pattern in the air-region. When the effective index of refraction in the magnetoplasma is imaginary which occurs at very low frequencies, the resulting radiation pattern is not symmetric at all. Moreover, in this case when the steady magnetic field reverses its orientation, the radiation pattern takes on a form that is exactly the mirror image about the vertical axis of the radiation pattern before the steady magnetic field

was reversed. When the effective index of refraction in the magnetoplasma is between zero and one, the radiation pattern is symmetric in the wedge region about the vertical axis bounded by the lines representing the critical angles but it is not symmetric between these lines and the interface. Finally, when the effective index of refraction in magnetoplasma is greater than one, there is no critical angle and the pattern is perfectly symmetric. The physical interpretation of these phenomena is as follows. For polar angles smaller than the critical angle, the transmitted rays undergo the usual refraction fixed by Snell's law for real angles. Now since the radiation from the source itself is symmetric, the resultant portion of the pattern in the air is also symmetric within the wedge region limited by the lines representing the critical angles. For the transmission angles beyond the critical angles, the Snell's law can be satisfied only by complex angles in the magnetoplasma which means that the transmitted energy in the air is produced by the inhomogeneous waves in the magnetoplasma which are also being emanated from the source. A combined action of the interface and the steady magnetic field destroys the symmetry of the inhomogeneous wave amplitudes in the magnetoplasma which then results in lack of symmetry of the radiation pattern in the air in the regions beyond the critical angles.

Although the problems treated in Part II and III are two-dimensional, they give a great deal of insight into the corresponding three-dimensional problems. The following correspondences can be easily deduced by physical reasoning:

(a) The radiation pattern from an electric current line source with H_{DC} normal to the line source is the same as it would result from a horizontal electric dipole in the plane normal to the dipole axis when H_{DC} is normal to the dipole axis.

Appendix A

EVALUATION OF THE INTEGRAL $\int_{-\pi}^{\pi} \frac{e^{i\lambda\rho\cos(\beta-\varphi)}}{\sin\beta}\, d\beta$

The integral under consideration is

$$I = \int_{-\pi}^{\pi} \frac{e^{i\lambda\rho\cos(\beta-\varphi)}}{\sin\beta}\, d\beta \; . \tag{A.1}$$

Expanding the numerator in terms of Bessel functions, (22, p. 372), gives

$$e^{i\lambda\rho\cos(\beta-\varphi)} = \sum_{n=-\infty}^{\infty} i^n J_n(\lambda\rho)\, e^{in(\beta-\varphi)} \tag{A.2}$$

Substituting this into (A.1) and interchanging the order of summation and integration, we obtain

$$I = \sum_{n=-\infty}^{\infty} i^n e^{-in\varphi} J_n(\lambda\rho) \int_{-\pi}^{\pi} \frac{e^{in\beta}}{\sin\beta}\, d\beta \; . \tag{A.3}$$

Now we focus our attention on the integral

$$I_1 = \int_{-\pi}^{\pi} \frac{e^{in\beta}}{\sin\beta}\, d\beta = \int_{0}^{2\pi} \frac{e^{in\beta}}{\sin\beta}\, d\beta \tag{A.4}$$

and make the transformation

$$z = e^{i\beta} \tag{A.5}$$

so that

$$d\beta = -i\,\frac{dz}{z}, \qquad \sin\beta = \frac{z^2-1}{2iz} \; . \tag{A.6}$$

Substituting the above results into (A.4) gives

$$I_1 = 2\int_C \frac{z^n\, dz}{(z+1)\,(z-1)} \tag{A.7}$$

where C is a unit circle.

Now we consider

$$K = \int_{\Gamma} \frac{z^n \, d\, z}{(z+1)(z-1)} \tag{A.8}$$

where Γ is the path in the complex z-plane as shown in Fig. A.1. Since the contour $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ contains no singularities, then by Cauchy's

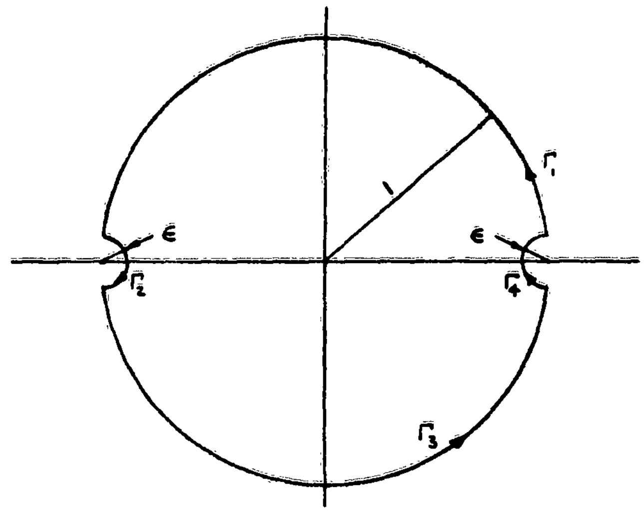

Figure A.1

theorem (15, p. 30)

$$K = \int_{\Gamma} \frac{z^n \, d\, z}{(z+1)\ (z-1)} = 0 = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4}$$

or (A.9)

$$\int_{\Gamma_1} + \int_{\Gamma_3} = -\int_{\Gamma_2} - \int_{\Gamma_4} \, .$$

Now

$$\int_{\Gamma_1} + \int_{\Gamma_3} = \int_{\epsilon}^{\pi-\epsilon} \frac{z^n \, dz}{(z+1)(z-1)} + \int_{\pi+\epsilon}^{2\pi-\epsilon} \frac{z^n \, dz}{(z+1)(z-1)} \tag{A.10}$$

$$\xrightarrow[\epsilon \longrightarrow 0]{} \int_{0}^{2\pi} \frac{z^n d\, z}{(z+1)(z-1)} = \frac{1}{2} I_1 \, .$$

Furthermore, on Γ_2 we put $z = -1 + \epsilon\, e^{-i\theta}$, $\pi/2 > \theta > -\pi/2$ and obtain

$$\int_{\Gamma_2} = -i\int_{\pi/2}^{-\pi/2} \frac{(-1+\epsilon e^{-i\theta})^n \epsilon e^{-i\theta}}{(2-\epsilon e^{-i\theta})\,\epsilon e^{-i\theta}}\, d\theta \xrightarrow[\epsilon \to 0]{} \frac{\pi i}{2}(-1)^n . \tag{A.11}$$

On Γ_4 we put $z = 1 + \epsilon e^{-i\theta}$, $-\pi/2 > \theta > -3\pi/2$ and obtain

$$\int_{\Gamma_4} = i\int_{-\pi/2}^{\pi/2} \frac{(1+\epsilon e^{-i\theta})^n \epsilon e^{-i\theta}}{(2+\epsilon e^{-i\theta})\,\epsilon e^{-i\theta}}\, d\theta \xrightarrow[\epsilon \to 0]{} -\frac{\pi i}{2} . \tag{A.12}$$

Thus, by (A.10), (A.11), and (A.12) we obtain

$$I = \pi i\left[\, 1 - (-1)^n \,\right] . \tag{A.13}$$

Now substituting (A.13) into (A.3) gives

$$I = \pi i \sum_{n=-\infty}^{\infty} i^n e^{-in\varphi}\left[1 - (-1)^n\right] J_n(\lambda\rho) . \tag{A.14}$$

For integer values of n the following relation is valid (22, p. 357)

$$J_{-n} = (-1)^n J_n . \tag{A.15}$$

Thus, we obtain finally

$$I = 4\pi i \sum_{n=0}^{\infty} (-1)^n \sin(2n+1)\varphi\, J_{2n+1}(\lambda\rho) . \tag{A.16}$$

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